15.6, 16.1 HW

11/2/20

15.6: # 1, 3, 13, 15, 17, 19, 23

1. Determine image under $G(u,v) = (2u, u+v)$ of following sets:

(a) u- & v-axes: $G(u,0) = (2u, u+0) = (2u, u)$ (u)

$G(0,v) = (0, 0+v) = (0, v)$ (v)

(b) rectangle $R = [0,5] \times [0,7]$: $G(0,0) = (2\cdot 0, 0+0) = (0,0)$

$G(5,0) = (2\cdot 5, 5+0) = (10,5)$

$G(5,7) = (2\cdot 5, 5+7) = (10,12)$

$G(0,7) = (2\cdot 0, 0+7) = (0,7)$

(c) line segment joining (1,2) & (5,3): $G(1,2) = (2\cdot 1, 1+2) = (2,3)$

$G(5,3) = (2\cdot 5, 5+3) = (10,8)$

(d) Δ w/ vertices (0,1), (1,0), & (1,1): $G(0,1) = (2\cdot 0, 0+1) = (0,1)$

$G(1,0) = (2\cdot 1, 1+0) = (2,1)$

$G(1,1) = (2\cdot 1, 1+1) = (2,2)$

3. $G(u,v) = (u^2, v)$

(a) u- & v-axes: $(x,y) = G(u,0) = (u^2, 0) \Rightarrow x = u^2, y = 0$ (u)

$(x,y) = G(0,v) = (0^2, v) = (0, v) \Rightarrow x = 0, y = v$ (v)

(b) rectangle $R = [-1,1] \times [-1,1]$: $|u| \leq 1$, $|v| \leq 1 \rightarrow x = u^2$ & $y = v$

$|\pm\sqrt{x}| \leq 1$, $|y| \leq 1 \leftarrow u = \pm\sqrt{x}$ & $v = y$

$\hookrightarrow 0 \leq x \leq 1$ & $-1 \leq y \leq 1$

(c) line segment joining (0,0) & (1,1): $0 \leq u \leq 1$, $v = u$

$0 \leq \sqrt{x} \leq 1$, $y = \sqrt{x}$

$0 \leq x \leq 1$, $y = \sqrt{x}$

(d) Δ w/ vertices $\overset{O}{(0,0)}$, $A(0,1)$, & $(1,1)_B$: $\overline{OA}$: $u = 0$ & $0 \leq v \leq 1$

$\pm\sqrt{x} = 0$ & $0 \leq y \leq 1$

$x = 0$, $0 \leq y \leq 1$

$\overline{AB}$: $0 \leq u \leq 1$ & $v = 1$

$0 \leq \sqrt{x} \leq 1$, $y = 1$

$0 \leq x \leq 1$, $y = 1$

$\overline{OB}$: (c) curve $y = \sqrt{x}$, $0 \leq x \leq 1$

13. $G(u,v) = (3u + 4v, u - 2v)$

$$\operatorname{Jac}(G) = \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} 3 & 4 \\ 1 & -2 \end{vmatrix} = 3\cdot(-2) - 1\cdot 4 = -6 - 4 = -10$$

15. $G(r,t) = (r\sin t,\ r-\cos t),\quad (r,t) = (1,\pi)$

$x = r\sin t$ & $y = r - \cos t$

$$\text{Jac}(G) = \frac{d(x,y)}{d(r,t)} = \begin{vmatrix} dx/dr & dx/dt \\ dy/dr & dy/dt \end{vmatrix} = \begin{vmatrix} \sin t & r\cos t \\ 1 & \sin t \end{vmatrix} = \sin^2 t - r\cos t$$

$\text{Jac}(G)(1,\pi) = \sin^2\pi - 1\cdot\cos\pi = 0 - 1\cdot(-1) = 1$

17. $G(r,\theta) = (r\cos\theta,\ r\sin\theta),\quad (r,\theta) = (4, \pi/6)$

$x = r\cos\theta$ & $y = r\sin\theta$

$$\text{Jac}(G) = \frac{d(x,y)}{d(r,\theta)} = \begin{vmatrix} dx/dr & dx/d\theta \\ dy/dr & dy/d\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\underbrace{\cos^2\theta + r\sin^2\theta}_{1} = r$$

$\text{Jac}(G)(4, \pi/6) = 4$

19. Find a linear mapping G that maps $[0,1]\times[0,1]$ to the $\square$ in the plane spanned by the vectors $\langle 2,3\rangle$ & $\langle 4,1\rangle$.

$G(u,v) = (Au + Cv,\ Bu + Dv)$ (Linear maps: $G(0,0) = (0,0)$)

$G(0,1) = (2,3),\quad G(1,0) = (4,1)$

$(A\cdot 0 + C\cdot 1,\ B\cdot 0 + D\cdot 1) = (C,D) = (2,3) \Rightarrow C = 2,\ D = 3$

$(A\cdot 1 + C\cdot 0,\ B\cdot 1 + D\cdot 0) = (A,B) = (4,1)\quad A = 4,\ B = 1$

$G(u,v) = (4u + 2v,\ u + 3v)$

23. $G(u,v) = (3u + v,\ u - 2v)$

$$\text{Jac } G = \frac{d(x,y)}{d(u,v)} = \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix} = -6 - 1 = -7$$

$\text{Area}(G(R)) = |\text{Jac } G|\,\text{Area}(R) = 7\cdot\text{Area}(R)$

(a) $R = [0,3]\times[0,5]$

$\text{Area}(R) = 3\cdot 5 = 15 \Rightarrow \text{Area}(G(R)) = 7\cdot 15 = 105$

(b) $R = [2,5]\times[1,7]$

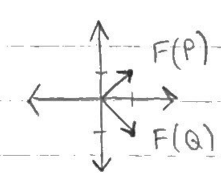
$\text{Area}(R) = 3\cdot 6 = 18 \Rightarrow \text{Area}(G(R)) = 7\cdot 18 = 126$

16.1: # 1-3, 5, 7, 9, 11, 17, 23, 25, 27

1. $P = (1,2),\ Q = (-1,-1)\ ;\ F = \langle x^2, x\rangle$

$F(1,2) = \langle 1^2, 1\rangle = \langle 1,1\rangle$

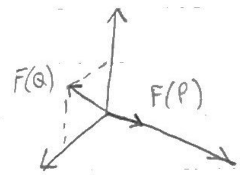
$F(-1,-1) = \langle (-1)^2, -1\rangle = \langle 1,-1\rangle$



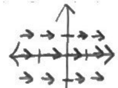
3. $P = (0,1,1)$, $Q = (2,1,0)$; $F = \langle xy, z^2, x \rangle$

$F(P) = \langle 0 \cdot 1 , 1^2 , 0 \rangle = \langle 0,1,0 \rangle$

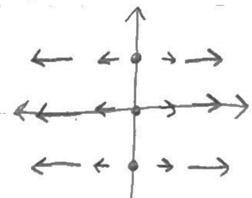
$F(Q) = \langle 2 \cdot 1 , 0^2 , 2 \rangle = \langle 2,0,2 \rangle$



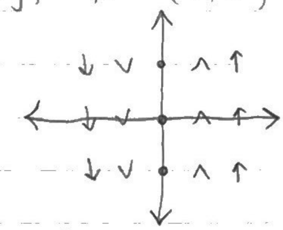
5. $F = \langle 1,0 \rangle$



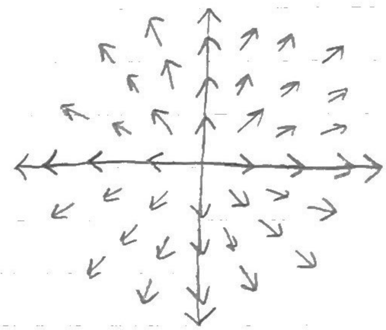
7. $F = xi \Rightarrow F(x,y) = xi = (x,0)$



9. $F = \langle 0, x \rangle$



11. $F = \left\langle \frac{x}{x^2+y^2} , \frac{y}{x^2+y^2} \right\rangle$



17. Plot (C)

23. $F = \langle x, y, z \rangle$

$\mathrm{div}(F) = \frac{d}{dx}(x) + \frac{d}{dy}(y) + \frac{d}{dz}(z) = 1+1+1 = 3$

$\mathrm{Curl}(F) = \left\langle \frac{dF_3}{dy} - \frac{dF_2}{dz} , \frac{dF_1}{dz} - \frac{dF_3}{dx} , \frac{dF_2}{dx} - \frac{dF_1}{dy} \right\rangle$

$= \left\langle \frac{dz}{dy} - \frac{dy}{dz} , \frac{dx}{dz} - \frac{dz}{dx} , \frac{dy}{dx} - \frac{dx}{dy} \right\rangle = \langle 0,0,0 \rangle$

25. $F = \langle x - 2zx^2 , z - xy , z^2x^2 \rangle$

$\mathrm{div}(F) = \frac{d}{dx}(x - 2zx^2) + \frac{d}{dy}(z - xy) + \frac{d}{dz}(z^2x^2) = (1-4zx) + (-x) + (2zx^2)$

$= 1 - 4zx - x + 2zx^2$

$\mathrm{curl}(F) = \langle 0-1 , -2x^2 - 2z^2x , -y-0 \rangle = \langle -1, -2x(x+z^2), -y \rangle$

27. $F = \langle yz , xz , xy \rangle$

$\mathrm{div}(F) = \frac{d}{dx}(yz) + \frac{d}{dy}(xz) + \frac{d}{dz}(xy) = 0+0+0 = 0$

$\mathrm{curl}(F) = \left\langle \frac{d}{dy}(xy) - \frac{d}{dz}(xz) , \frac{d}{dz}(yz) - \frac{d}{dx}(xy) , \frac{d}{dx}(xz) - \frac{d}{dy}(yz) \right\rangle$

$= \langle 0,0,0 \rangle$