HW due 11/22/20

16.4: 7, 13, 15, 19

16.5: 5, 7, 9, 11

16.4

7. $G(u,v) = (2u+v,\ u-4v,\ 3u) \quad u=1 \quad v=4$

$T_u = \langle 2,1,3\rangle \qquad T_v = \langle 1,-4,0\rangle$

$$N(u,v) = T_u \times T_v = \begin{vmatrix} i & j & k \\ 2 & 1 & 3 \\ 1 & -4 & 0 \end{vmatrix} = i\big((1)(0)-(3)(-4)\big) - j\big((2)(0)-(3)(1)\big) + k\big((2)(-4)-(1)(1)\big)$$

$$= 12i + 3j - 9k = \boxed{3\langle 4,1,-3\rangle}$$

$G(1,4) = (6,-15,3)$

$\langle x-6,\ y+15,\ z-3\rangle \cdot 3\langle 4,1,-3\rangle = 0$

$4(x-6) + (y+15) - 3(z-3) = 0$

$4x - 24 + y + 15 - 3z + 9 = 0$

$\boxed{4x+y-3z=0}$

13. $G(u,v) = (u\cos v,\ u\sin v,\ u) \quad 0 \le u \le 1 \quad 0 \le v \le 1 \quad f(x,y,z) = z(x^2+y^2)$

$\iint_S f(x,y,z)\,dS = \int_a^b \int_c^d f(G(u,v))\|N\|\,du\,dv$

$f(u\cos v,\ u\sin v,\ u) = u(u^2\cos^2 v + u^2\sin^2 v)$

$= u^3(\cos^2 v + \sin^2 v) = u^3$

$G_u = (\cos v,\ \sin v,\ 1) \qquad G_v = (-u\sin v,\ u\cos v,\ 0)$

$$N = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = i(0-u\cos v) - j(0+u\sin v) + k(u\cos^2 v + u\cos^2 v)$$

$= -u\cos v\, i - u\sin v\, j + uk$

$\|N\| = \sqrt{(-u\cos v)^2 + (-u\sin v)^2 + u^2} = \sqrt{u^2(\cos^2 v + \sin^2 v) + u^2} = \sqrt{2u^2}$

$= u\sqrt{2}$

$\int_0^1\int_0^1 u^3 u(\sqrt{2})\,du\,dv = \int_0^1\int_0^1 \sqrt{2}\,u^4\,du\,dv$

$= \frac{\sqrt{2}}{6}u^5\Big|_0^1 = \frac{\sqrt{2}}{5}$

$= \frac{\sqrt{2}}{6}v\Big|_0^1 = \boxed{\frac{\sqrt{2}}{6}}$

15. $y = 9 - z^2 \quad 0 \le x \le 3 \quad 0 \le z \le 3 \quad f(x,y,z) = z$

$G(u,v) = \langle u,\ 9-v^2,\ v\rangle \quad 0 \le u \le 3 \quad 0 \le v \le 3$

$T_u = \langle 1,0,0\rangle \qquad T_v = \langle 0,-2v,1\rangle$

$$N = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & -2v & 1 \end{vmatrix} = i(0-0) - j(1-0) + k(-2v-0)$$

$= \langle 0,-1,-2v\rangle$

$\|N\| = \sqrt{0^2 + (-1)^2 + (2v)^2} = \sqrt{4v^2+1}$

15 (cont.). $f(u,\ 9-v^2,\ v) = v$

$\int_0^3\int_0^3 v\sqrt{4v^2+1}\,du\,dv$

$= uv\sqrt{4v^2+1}\Big|_0^3 = 3v\sqrt{4v^2+1}$

$1+4v^2 = t^2 \qquad 8v\,dv = 2t\,dt$

$v\,dv = \frac{1}{4}t\,dt \qquad v=0 \quad t=1 \qquad v=3 \quad t=\sqrt{37}$

$\int_1^{\sqrt{37}} \frac{3}{4}t^2\,dt = \frac{t^3}{4}\Big|_1^{\sqrt{37}} = \frac{37\sqrt{37}}{4} - \frac{1}{4} = \boxed{9\sqrt{37}}$

19. $x^2+y^2 = 4 \quad 0 \le z \le 4 \quad f(x,y,z) = e^{-z}$

$G(\theta,z) = (2\cos\theta,\ 2\sin\theta,\ z) \quad 0 \le \theta \le 2\pi \quad 0 \le z \le 4$

$T_\theta = \langle -2\sin\theta,\ 2\cos\theta,\ 0\rangle \qquad T_z = \langle 0,0,1\rangle$

$$N = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = i(2\cos\theta - 0) - j(-2\sin\theta - 0) + k(0-0)$$

$= \langle 2\cos\theta,\ 2\sin\theta,\ 0\rangle$

$\|N\| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2 + 0^2} = \sqrt{4\cos^2\theta + 4\sin^2\theta} = \sqrt{4} = 2$

$f(2\cos\theta,\ 2\sin\theta,\ z) = e^{-z}$

$\int_0^{2\pi}\int_0^4 2e^{-z}\,dz\,d\theta = -2e^{-z}\Big|_0^4 = (-2e^{-4}+2)$

$2(1-e^{-4})\theta\Big|_0^{2\pi} = \boxed{4\pi(1-e^{-4})}$

16.5

5. $F = \langle y,z,x\rangle \qquad 3x - 4y + z = 1$

$G(x,y) = \langle x,\ y,\ 1-3x+4y\rangle \quad 0 \le x,\ y \le 1$

$T_x = \langle 1,0,-3\rangle \qquad T_y = \langle 0,1,4\rangle$

$$N = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & 4 \end{vmatrix} = i(0+3) - j(4-0) + k(1-0)$$

$= 3i - 4j + 1k = \langle 3,-4,1\rangle$

$F(x,y,z) = (y,z,x) = \langle y,\ 1-3x+4y,\ x\rangle$

$\langle y,\ 1-3x+4y,\ x\rangle \cdot \langle 3,-4,1\rangle = 3y - 4 + 12x - 16y + x$

$= 13x - 13y - 4$

$\int_0^1\int_0^1 13x - 13y - 4\,dx\,dy = \frac{13x^2}{2} - 13xy - 4x\Big|_0^1$

$= \int_0^1 \frac{13}{2} - 13y - 4\,dy = \frac{5}{2}y - \frac{13y^2}{2}\Big|_0^1 = \frac{5}{2} - \frac{13}{2} = -\frac{8}{2} = \boxed{-4}$

7. $F = \langle 0, 3, x \rangle \quad x^2 + y^2 + z^2 = 9 \quad x \geq 0 \quad y \geq 0 \quad z \geq 0$

$G(\theta, \phi) = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi) \quad 0 \leq \theta \leq \frac{\pi}{2} \quad 0 \leq \phi \leq \frac{\pi}{2}$

$T_\theta = \langle -3\sin\theta\sin\phi, 3\cos\theta\sin\phi, 0 \rangle$

$T_\phi = \langle 3\cos\theta\cos\phi, 3\sin\theta\cos\phi, -3\sin\phi \rangle$

$$N = \begin{vmatrix} i & j & k \\ -3\sin\theta\sin\phi & 3\cos\theta\sin\phi & 0 \\ 3\cos\theta\cos\phi & 3\sin\theta\cos\phi & -3\sin\phi \end{vmatrix}$$

$= -9\cos\theta\sin^2\phi\, i - 9\sin\theta\sin^2\phi\, j - 9\sin\phi\cos\phi\, k$

$N = 9\cos\theta\sin^2\phi\, i + 9\sin\theta\sin^2\phi\, j + 9\sin\phi\cos\phi\, k$

$F(G(\theta, \phi)) = \langle 0, 3, 3\cos\theta\sin\phi \rangle$

$\int_0^{\pi/2}\int_0^{\pi/2} \langle 0, 3, 3\cos\theta\sin\phi \rangle \cdot \langle 9\cos\theta\sin^2\phi, 9\sin\theta\sin^2\phi, 9\sin\phi\cos\phi \rangle$

$= \iint (27\sin\theta\sin^2\phi + 27\cos\theta\sin^2\phi\cos^2\phi)\, d\theta\, d\phi$

$= 27\int_0^{\pi/2} \sin\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\, d\phi + 27\int_0^{\pi/2} \cos\theta\, d\theta \int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi$

$= -27\cos\theta \Big|_0^{\pi/2} \int_0^{\pi/2} \sin^2\phi\, d\phi + 27\sin\theta \Big|_0^{\pi/2} \int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi$

$= \frac{27}{2}\left(\phi - \frac{\sin 2\phi}{2}\right)\Big|_0^{\pi/2} + 27\int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi$

$= \frac{27\pi}{4} + 27\int_0^{\pi/2} \sin^2\phi\cos\phi\, d\phi$

$w = \sin\phi \quad w = 0 \text{ to } 1$

$\frac{27\pi}{4} + 27\int_0^1 w^2\, dw = \frac{27\pi}{4} + 27\frac{w^3}{3}\Big|_0^1 = \boxed{\frac{27\pi}{4} + 9}$

9. $F = \langle z, z, x \rangle \quad z = 9 - x^2 - y^2 \quad x \geq 0 \quad y \geq 0 \quad z \geq 0$

$G(r, \theta) = (r\cos\theta, r\sin\theta, 9 - r^2) \quad 0 \leq r \leq 3 \quad 0 \leq \theta \leq \frac{\pi}{2}$

$T_r = \langle \cos\theta, \sin\theta, -2r \rangle \quad T_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$

$$N = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = 2r^2\cos\theta\, i + 2r^2\sin\theta\, j + r\, k$$

$F(G(r, \theta)) = \langle 9 - r^2, 9 - r^2, r\cos\theta \rangle$

$\iint (2r^2(9 - r^2)(\cos\theta + \sin\theta) + r^2\cos\theta)\, dr\, d\theta$

$= \int_0^3 2r^2(9 - r^2)\, dr \int_0^{\pi/2} (\cos\theta + \sin\theta)\, d\theta + \int_0^3 r^2\, dr \int_0^{\pi/2} \cos\theta\, d\theta$

$= \left(6r^3 - \frac{2r^5}{5}\right)\Big|_0^3 \times (\sin\theta - \cos\theta)\Big|_0^{\pi/2} + \frac{r^3}{3}\Big|_0^3 \times \sin\theta\Big|_0^{\pi/2}$

$= 27\left(6 - \frac{18}{5}\right)(1 + 1) + 9(1 - 0)$

$= 54\left(\frac{12}{5}\right) + 9 = \boxed{\frac{693}{5}}$

11. $F = y^2 i + z j - x k \quad x + y + z = 1 \quad x, y, z \geq 0$

$x = u \quad y = v \quad z = 1 - u - v \quad G(u, v) = (u, v, 1 - u - v)$

$F(G(u, v)) = \langle v^2, 2, -u \rangle$

$T_u = \langle 1, 0, -1 \rangle \quad T_v = \langle 0, 1, -1 \rangle$

$$N = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (0 + 1)i - (-1 - 0)j + (1 - 0)k = i + j + k = \langle 1, 1, 1 \rangle$$

11 (cont.) $\iint v^2 + 2 - u\, du\, dv$

$v^2 u + 2u - \frac{u^2}{2}\Big|_0^{1-v} = (1 - v)v^2 + 2(1 - v) - \frac{(1 - v)^2}{2}$

$\int_0^1 \left(v^2 - v^3 + 2 - 2v - \frac{(1 - v)^2}{2}\right) dv$

$= \frac{v^3}{3} - \frac{v^4}{4} + 2v - v^2 + \frac{(1 - v)^3}{6}\Big|_0^1$

$= \frac{1}{3} - \frac{1}{4} + 2 - 1 - \frac{1}{6} = \boxed{\frac{11}{12}}$