16.4 - 16.5 HW

11/16/20

16.4 : 7, 13, 15, 19

7. $\phi(u,v) = (2u+v,\ u-4v,\ 3u)$; $u=1$, $v=4$

$T_u = d\phi/du = d/du\,(2u+v,\ u-4v,\ 3u) = \langle 2,1,3\rangle$

$T_v = d\phi/dv = d/dv\,(2u+v,\ u-4v,\ 3u) = \langle 1,-4,0\rangle$

$N(u,v) = T_u \times T_v = \begin{vmatrix} i & j & K \\ 2 & 1 & 3 \\ 1 & -4 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -4 & 0 \end{vmatrix} i - \begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ 1 & -4 \end{vmatrix} K$

$= 12i + 3j - 9K = 3\langle 4,1,-3\rangle$

$P: \phi(1,4) = (6,-15,3)$; normal vector : $\langle 4,1,-3\rangle$

$\langle x-6,\ y+15,\ z-3\rangle \cdot \langle 4,1,-3\rangle = 0$

$4(x-6) + y+15 - 3(z-3) = 0 \quad \Rightarrow \quad 4x+y-3z = 0$

13. $\phi(u,v) = (u\cos v,\ u\sin v,\ u)$, $0 \le u \le 1$, $0 \le v \le 1$; $f(x,y,z) = z(x^2+y^2)$

S1. Compute tangent & normal vectors

$T_u = d\phi/du = d/du\,(u\cos v,\ u\sin v,\ u) = \langle \cos v,\ \sin v,\ 1\rangle$

$T_v = d\phi/dv = d/dv\,(u\cos v,\ u\sin v,\ u) = \langle -u\sin v,\ u\cos v,\ 0\rangle$

$n = T_u \times T_v = \begin{vmatrix} i & j & K \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0 \end{vmatrix} = (-u\cos v)i - (u\sin v)j + (u\cos^2 v + u\sin^2 v)K$

$= (-u\cos v)i - (u\sin v)j + uK = \langle -u\cos v,\ -u\sin v,\ u\rangle$

$\|N\| = \sqrt{(-u\cos v)^2 + (-u\sin v)^2 + u^2} = \sqrt{u^2(\cos^2 v + \sin^2 v + 1)} = \sqrt{u^2 \cdot 2} = \sqrt{2}\,|u| = \sqrt{2}u$

S2. Calc. Surface integral

$f(\phi,(u,v)) = u(u^2\cos^2 v + u^2\sin^2 v) = u \cdot u^2 = u^3$

$\iint_S f(x,y,z)\,dS = \int_0^1\int_0^1 f(\phi,(u,v))\,\|N\|\,du\,dv = \int_0^1\int_0^1 u^3 \cdot \sqrt{2}u\,du\,dv$

$= \int_0^1 \sqrt{2}\,dv = \int_0^1 u^4\,du = \sqrt{2} \cdot \frac{u^5}{5}\Big|_0^1 = \frac{\sqrt{2}}{5}$

15. $y = 4-z^2$, $0 \le x \le 2$, $0 \le z \le 2$; $f(x,y,z) = 3z$

Let $y = g(x,z) = 4-z^2$. Then $g_x = 0$ & $g_z = -2z$, so that

$dS = \sqrt{1+g_x^2+g_z^2} = \sqrt{4z^2+1}$

$\iint_D f(x,y,z)\,dS = \int_0^2\int_0^2 3z\sqrt{4z^2+1}\,dx\,dz = \int_0^2 6z\sqrt{4z^2+1}\,dz$

$= \frac{3}{4}\int_0^2 8z\sqrt{4z^2+1}\,dz = \frac{3}{4}\cdot\frac{2}{3}(4z^2+1)^{3/2}\Big|_0^2 = \frac{1}{2}(17\sqrt{17}-1)$

19. $x^2+y^2=4$, $0 \le z \le 4$; $f(x,y,z) = e^{-z}$

$\phi(\theta, z) = (2\cos\theta, 2\sin\theta, z)$, $0 \le \theta \le 2\pi$, $0 \le z \le 4$

S1. $T_\theta = d\phi/d\theta = d/d\theta\,(2\cos\theta, 2\sin\theta, z) = \langle -2\sin\theta, 2\cos\theta, 0 \rangle$

$T_z = d/dz\,(2\cos\theta, 2\sin\theta, z) = \langle 0, 0, 1 \rangle$

$N(\theta, z) = T_\theta \times T_z = \begin{vmatrix} i & j & k \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2\cos\theta)i + (2\sin\theta)j = \langle 2\cos\theta, 2\sin\theta, 0 \rangle$

$\|N(\theta,z)\| = \sqrt{(2\cos\theta)^2 + (2\sin\theta)^2 + 0} = \sqrt{4(\cos^2\theta + \sin^2\theta)} = \sqrt{4} = 2$

S2. $\iint_S f(x,y,z)\,dS = \iint_D f(\phi(\theta,z))\,\|N\|\,d\theta\,dz = \int_0^{2\pi}\int_0^4 e^{-z} \cdot 2\,d\theta\,dz$

$= \int_0^{2\pi} 2\,d\theta \int_0^4 e^{-z}\,dz = 4\pi \cdot (-e^{-z}) \Big|_0^4 = 4\pi(1 - e^{-4})$

16.5: 5, 7, 9, 11

5. $F = \langle y, z, x \rangle$, plane $3x - 4y + z = 1$, $0 \le x \le 1$, $0 \le y \le 1$, upward-pointing normal

$z = 1 - 3x + 4y \Rightarrow \phi(x,y) = (x, y, 1-3x+4y)$

S1. $T_x = d\phi/dx = d/dx\,(x, y, 1-3x+4y) = \langle 1, 0, -3 \rangle$

$T_y = d\phi/dy = d/dy\,(x, y, 1-3x+4y) = \langle 0, 1, 4 \rangle$

$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -3 \\ 0 & 1 & 4 \end{vmatrix} = 3i - 4j + k = \langle 3, -4, 1 \rangle \Rightarrow N = \langle 3, -4, 1 \rangle$

S2. Evaluate dot product $F \cdot N$

$F(\phi(x,y)) = \langle y, z, x \rangle = \langle y, 1-3x+4y, x \rangle$

$F(\phi(x,y)) \cdot N = \langle y, 1-3x+4y, x \rangle \cdot \langle 3, -4, 1 \rangle = 3y - 4(1-3x+4y) + x$

$= 13x - 13y - 4$

S3. Evaluate surface integral

$\iint_S F \cdot ds = \iint_D F(\phi(x,y)) \cdot N(x,y)\,dx\,dy = \int_0^1\int_0^1 (13x - 13y - 4)\,dx\,dy$

$= \int_0^1 \frac{13x^2}{2} - 13yx - 4x \Big|_{x=0}^{1} dy = \int_0^1 \frac{13}{2} - 13y - 4\,dy = \frac{5y}{2} - \frac{13y^2}{2} \Big|_0^1 = -4$

7. $F = \langle 0, 3, x \rangle$, part of sphere $x^2 + y^2 + z^2 = 9$, where $x \ge 0$, $y \ge 0$, $z \ge 0$, outward-pointing normal

$\phi(\theta, \phi) = (3\cos\theta\sin\phi, 3\sin\theta\sin\phi, 3\cos\phi)$; $0 \le \theta \le \pi/2$, $0 \le \phi \le \pi/2$

S1. Compute normal vector

$N = T_\phi \times T_\theta = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$

S2. Evaluate $F \cdot N$

$F(\Phi(\theta,\phi)) = \langle 0, 3, x \rangle = \langle 0, 3, 3\cos\theta \sin\phi \rangle$

$$F(\Phi(\theta,\phi)) \cdot N(\theta,\phi) = \langle 0, 3, 3\cos\theta \sin\phi \rangle \cdot \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$$
$$= \sin\phi \,(3\sin\theta \sin\phi + 3\cos\theta \sin\phi \cos\phi)$$
$$= 3\sin\theta \sin^2\phi + 3\cos\theta \sin^2\phi \cos\phi$$

S3. Evaluate surface integral

$$\iint_S F \cdot dS = \iint_D F(\Phi(\theta,\phi)) \cdot N(\theta,\phi)\, d\theta\, d\phi$$
$$= \int_0^{\pi/2}\int_0^{\pi/2} (3\sin\theta \sin^2\phi + 3\cos\theta \sin^2\phi \cos\phi)\, d\theta\, d\phi$$
$$= \int_0^{\pi/2} (-3\cos\theta \sin^2\phi + 3\sin\theta \sin^2\phi \cos\phi) \Big|_{\theta=0}^{\pi/2} d\phi$$
$$= \int_0^{\pi/2} (3\sin^2\phi + 3\sin^2\phi \cos\phi)\, d\phi = \int_0^{\pi/2} \frac{3}{2} - \frac{3}{2}\cos 2\phi + 3\sin^2\phi \cos\phi \, d\phi$$
$$= \frac{3}{2}\phi - \frac{3}{4}\sin 2\phi + \sin^3\phi \Big|_0^{\pi/2} = \frac{3\pi}{4} + 1$$

9. $F = \langle z, z, x \rangle$, $z = 9 - x^2 - y^2$, $x \ge 0$, $y \ge 0$, $z \ge 0$, upward-pointing normal

S1. Find parametrization

$\phi(x,y) = (x, y, 9 - x^2 - y^2)$, $x \ge 0$, $y \ge 0$, $z \ge 0$

$D = \{(x,y) : x^2 + y^2 \le 9;\ x \ge 0,\ y \ge 0\}$

S2. Compute tan. & normal vectors

$T_x = d\phi/dx = d/dx\,(x, y, 9 - x^2 - y^2) = \langle 1, 0, -2x \rangle$

$T_y = d\phi/dy = d/dy\,(x, y, 9 - x^2 - y^2) = \langle 0, 1, -2y \rangle$

$$T_x \times T_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = (2x)i + (2y)j + k = \langle 2x, 2y, 1 \rangle \Rightarrow N = \langle 2x, 2y, 1 \rangle$$

S3. Evaluate $F \cdot N$

$F(\phi(x,y)) = \langle z, z, x \rangle = \langle 9 - x^2 - y^2, 9 - x^2 - y^2, x \rangle$

$$F(\phi(x,y)) \cdot N(x,y) = \langle 9 - x^2 - y^2, 9 - x^2 - y^2, x \rangle \cdot \langle 2x, 2y, 1 \rangle$$
$$= 2x(9 - x^2 - y^2) + 2y(9 - x^2 - y^2) + x$$

S4. Evaluate surface integral

$$\iint_S F \cdot dS = \iint_D F(\phi(x,y) \cdot N(x,y) dx\,dy$$

$$= \iint_D \left(2x(9-x^2-y^2) + 2y(9-(x^2+y^2)) + x\right) dx\,dy$$

$$\iint_S F \cdot dS = \int_0^3 \int_0^{\pi/2} \left(2r\cos\theta \cdot (9-r^2) + 2r\sin\theta \cdot (9-r^2) + r\cos\theta\right) \cdot r\,d\theta\,dr$$

$$= \int_0^3 \int_0^{\pi/2} \left((2r^2(9-r^2))(\cos\theta + \sin\theta) + r^2\cos\theta\right) d\theta\,dr$$

$$= \int_0^3 \left((2r^2(9-r^2))(\sin\theta - \cos\theta) + r^2\sin\theta\right)\Big|_0^{\pi/2} dr$$

$$= \int_0^3 \left(4r^2(9-r^2) + r^2\right) dr = \int_0^3 (37r^2 - 4r^4)\,dr = \frac{37}{3}r^3 - \frac{4}{5}r^5\Big|_0^3 = \frac{693}{5}$$

11. $F = y^2 i + 2j - xK$, portion of the plane $x+y+z=1$ in the octant $x,y,z \geq 0$, upward-pointing normal

$\phi(x,y) = (x, y, 1-x-y) \Rightarrow$ S1. $T_x = d\phi/dx = d/dx\,(x, y, 1-x-y) = \langle 1, 0, -1 \rangle$

S2. $F \cdot N$

$T_y = d\phi/dy = d/dy\,(x, y, 1-x-y) = \langle 0, 1, -1 \rangle$

$F(\phi(x,y)) \cdot N$

$T_x \times T_y = \begin{vmatrix} i & j & K \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = i + j + K = \langle 1,1,1 \rangle = N$

$= \langle y^2, 2, -x \rangle \cdot \langle -1,-1,-1 \rangle = -y^2 - 2 + x$

S3. $$\iint_S F \cdot dS = \iint_D F(\phi(x,y)) \cdot N\,dx\,dy = \int_0^1 \int_0^{1-y} (-y^2 - 2 + x)\,dx\,dy$$

$$= \int_0^1 -y^2x - 2x + \frac{x^2}{2}\Big|_{x=0}^{1-y} dy = \int_0^1 -y^2(1-y) - 2(1-y) + \frac{(1-y)^2}{2}\,dy$$

$$= \int_0^1 y^3 - y^2 + 2(y-1) + \frac{(y-1)^2}{2}\,dy = \frac{y^4}{4} - \frac{y^3}{3} + (y-1)^2 + \frac{(y-1)^3}{6}\Big|_0^1$$

$$= \frac{1}{4} - \frac{1}{3} - 1 - \frac{1}{6} = -\frac{11}{12}$$