16.2 : 3, 9, 11, 13, 17, 27, 29, 31, 35

3) Let $F=\langle y^2, x^2\rangle$, and let C be the curve $y=x^{-1}$ for $1\le x\le 2$, left to right

a) $F(r(t))$ $\quad dr = r'(t)$ $\quad r(t)=(t, t^{-1})$

$x=t$

$y=t^{-2}$ $\quad x^2=t^2$ $\quad F(r(t)) = \langle t^{-2}, t^2\rangle$ $\quad dr = \langle 1, -t^{-2}\rangle\, dt$

b) Calculate the dot product $F(r(t))\cdot r'(t)\,dt$ and evaluate $\int_C F\cdot dr$

$\langle t^{-2}, t^2\rangle \cdot \langle 1, -t^{-2}\rangle = t^{-2} - t^{-4} + t^2 - 1$
$= t^{-1} - 1$

$\int_1^2 t^{-1} - 1\, dt = -\frac{1}{2}$

9) $\int_C f\,ds$ for $f(x,y)=\sqrt{1+9xy}$, $y=x^3$ for $0\le x\le 1$

$x=t$ $\quad y=t^3$

$dx=1$ $\quad dy=3t^2$

$ds=\sqrt{1+9t^4}\,dt$

$\int_0^1 (\sqrt{1+9t^4})(\sqrt{1+9t^4})dt = \int_0^1 1+9t^4\,dt = t+\frac{9}{5}t^5\Big|_0^1 = 1+\frac{9}{5} = \frac{14}{5} = 2.8$

11) $\int_C f\,ds$ for $f(x,y,z)=z^2$, $r(t)=(2t, 3t, 4t)$ for $0\le t\le 2$

$x=2t$ $\quad y=3t$ $\quad z=4t$

$dx=2$ $\quad dy=3$ $\quad dz=4$

$s=\sqrt{4+9+16} = \sqrt{29}\,dt$

$\int_0^2 16t^2\sqrt{29}\,dt = \frac{16\sqrt{29}}{3}t^3\Big|_0^2 = \frac{128\sqrt{29}}{3} \approx 229.8$

13) $\int_C f\,ds$ for $f(x,y,z)=xe^{z^2}$ piecewise linear path from $(0,0,1)$ to $(0,2,0)$ to $(1,1,1)$

$PQ = (0,0,1) + t(0,2,-1)$
$= (0, 2t, 1-t)$

$x=0$ $\quad y=2t$ $\quad z=1-t$

$dx=0$ $\quad dy=2$ $\quad dz=-1$

$ds=\sqrt{5}\,dt$

$\int_0^1 0\sqrt{5}\,dt = 0$

$QR = (0,2,0) + t(1,-1,1)$
$= (t, 2-t, t)$

$x=t$ $\quad y=2-t$ $\quad z=t$

$dx=1$ $\quad dy=-1$ $\quad dz=1$

$ds=\sqrt{3}\,dt$

$\int_0^1 te^0\sqrt{3}\,dt$
$= \frac{\sqrt{3}}{2}t^2\Big|_0^1 = \frac{\sqrt{3}}{2}$

$RP = (1,1,1) + t(-1,-1,0)$
$= (1-t, 1-t, 1)$

$x=1-t$ $\quad y=1-t$ $\quad z=1$

$dx=-1$ $\quad dy=-1$ $\quad dz=0$

$ds=\sqrt{2}\,dt$

$\int_0^1 (1-t)e^1\sqrt{2}\,dt$
$= \sqrt{2}et - \frac{\sqrt{2}e}{2}t^2\Big|_0^1$
$= \sqrt{2}e - \frac{\sqrt{2}e}{2} = \frac{\sqrt{2}e}{2}$

$= \frac{\sqrt{3}}{2}(e-1)$

17) $\int_C 1\,ds$ $\quad r(t)=(4t, -3t, 12t)$ for $2\le t\le 5$

$x=4t$ $\quad y=-3t$ $\quad z=12t$

$dx=4$ $\quad dy=-3$ $\quad dz=12$

$ds=\sqrt{16+9+144}\,dt$
$= \sqrt{169}\,dt = 13\,dt$

$\int_2^5 13\,dt = 13t\Big|_2^5 = 39$

The integral represents the distance between $(8,-6,24)$ and $(20,-15,60)$

27) $\int_C y\,dx - x\,dy$ parabola $y=x^2$ for $0 \le x \le 2$

$x=t \quad y=t^2$
$dx=1 \quad dy=2t$

$\int_0^2 t^2(1) - t(2t)\,dt = \int_0^2 t^2 - 2t^2\,dt = \int_0^2 -t^2\,dt \qquad -\frac{t^3}{3}\Big|_0^2 = -\frac{8}{3}$

29) $\int_C (x-y)\,dx + (y-z)\,dy + z\,dz$, line segment from $(0,0,0)$ to $(1,4,4)$

$PQ = (0,0,0) + t(1,4,4) = (t, 4t, 4t) \qquad x=t \quad y=4t \quad z=4t$
$dx=1 \quad dy=4 \quad dz=4$

$\int_0^1 (t-4t)1 + (4t-4t)4 + 4t(4) = \int_0^1 -3t + 16t\,dt = \int_0^1 13t\,dt = \frac{13}{2}t^2\Big|_0^1 = \frac{13}{2}$

31) $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$, segment from $(1,0)$ to $(0,1)$

$PQ = (1,0) + t(-1,1) = (1-t, t) \qquad x=1-t \quad y=t \qquad dx=-1 \quad dy=1$

$\int_0^1 \frac{-t(-1) + (1-t)(1)}{(1-t)^2 + t^2}\,dt = \int_0^1 \frac{t + 1 - t}{1-2t+t+t^2}\,dt = \int_0^1 \frac{1}{1-t+t^2} = \frac{2\pi}{3\sqrt{3}} \qquad \frac{\pi}{2}$

35) $F(x,y,z) = \langle e^z, e^{x-y}, e^y \rangle$

$P=(0,0,0) \qquad R=(0,0,1) \qquad S=(0,1,1) \qquad Q=(-1,1,1)$

$PR = (0,0,0) + t(0,0,1)$
$= (0,0,t)$
$x=0 \quad y=0 \quad z=t$
$dx=0 \quad dy=0 \quad dz=1$
$ds = dt$

$RS = (0,0,1) + t(0,1,1)$
$= (0, t, 1+t)$
$x=0 \quad y=t \quad z=1+t$
$dx=0 \quad dy=1 \quad dz=1$
$ds = \sqrt{2}\,dt$

$SQ = (0,1,1) + t(-1,1,1)$
$= (-t, 1+t, 1+t)$
$x=-t \quad y=1+t \quad z=1+t$
$dx=-1 \quad dy=1 \quad dz=1$
$ds = \sqrt{3}\,dt$

$(P,Q,R) = (e^t, \cancel{e^0}\,1, \cancel{e^0}\,1)$

(e^{1+t}, e^{-t}, e^t)

$(e^{1+t}, e^{-2t-1}, e^{1+t})$

$\int_0^1 P\,dx + Q\,dy + R\,dz \qquad \int_0^1 P\,dx + Q\,dy + R\,dz \qquad \int_0^1 P\,dx + Q\,dy + R\,dz$

$\int_0^1 e^t(0) + 1(0) + 1(1)\,dt$

$\int_0^1 1\,dt = t\Big|_0^1 = 1$

$\int_0^1 e^{1+t}(0) + e^{-t}(1) + e^t(1)\,dt$

$\int_0^1 e^{-t} + e^t\,dt = e^{-\frac{t^2}{2}} + e^{\frac{t^2}{2}}\Big|_0^1$
$= e^{-\frac{1}{3}} + e^{\frac{1}{3}} =$

$\int_0^1 e^{1+t}(-1) + e^{-2t-1}(1) + e^{1+t}(1)\,dt$

$\int_0^1 -e^{1+t} + e^{-2t-1} + e^{1+t}\,dt$

$2 - e - \frac{1}{e}$

16.3 : 1, 3, 5, 9, 13, 15, 17, 19

1) Let $f(x,y,z) = xy\sin(yz)$ and $F = \nabla f$. Evaluate $\int_C F \cdot dr$ where C is any path from $(0,0,0)$ to $(1,1,\pi)$

$x = t \quad y = t \quad z = \pi t$
$dx = 1 \quad dy = 1 \quad dz = \pi$

$F = \text{grad}(f) = \langle y\sin(yz), x\cos(yz), xy\cos(yz) \rangle$

$t\sin\pi + t\cos\pi + \pi t\cos\pi \Big|_0^1 = 0 - 1 + 1 = 0$

$\int_0^1 \langle \sin(\pi), \cos(\pi), \cos(\pi) \rangle \cdot \langle dt, dt, \pi dt \rangle$

3) Verify that $F = \nabla f$ and evaluate the line integral of F over the given path

$F(x,y) = \langle 3, 6y \rangle$, $f(x,y) = 3x + 3y^2$; $r(t) = \langle t, 2t^{-1} \rangle$ for $1 \le t \le 4$

$F = \langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f) \rangle = \langle 3, 6y \rangle$ ✓

$x = t \quad y = 2/t$
$dx = 1dt \quad dy = -2t^{-2}dt$

$\int_1^4 \langle 3, 6y \rangle \cdot \langle dx, dy \rangle = \int_1^4 \langle 3, 12/t \rangle \cdot \langle dt, -2t^{-2}dt \rangle$

$= \int_1^4 \langle 3dt, -24t^{-3}dt \rangle = 3t + 12t^{-2} \Big|_1^4 = (12 + \frac{12}{16}) - (3 + 12) = \frac{12 - 3(16)}{16} = \frac{-36}{16} = -\frac{9}{4}$

5) Verify that $F = \nabla f$ and evaluate the line integral of F over the given path

$F(x,y,z) = ye^z i + xe^z j + xye^z k$, $f(x,y,z) = xye^z$ $\quad r(t) = (t^2, t^3, t-1)$ for $1 \le t \le 2$

$F = \langle \frac{\partial}{\partial x}(f), \frac{\partial}{\partial y}(f), \frac{\partial}{\partial z}(f) \rangle = \langle ye^z, xe^z, xye^z \rangle$ ✓

$x = t^2 \quad y = t^3 \quad z = t-1$
$dy = 2t\,dt \quad dy = 3t^2dt \quad dz = dt$

$\int_1^2 \langle ye^z, xe^z, xye^z \rangle \cdot \langle dx, dy, dz \rangle = \int_1^2 \langle t^3e^{t-1}, t^2e^{t-1}, t^5e^{t-1} \rangle \cdot \langle 2tdt, 3t^2dt, dt \rangle$

$= \int_1^2 \langle 2t^4e^{t-1}dt, 3t^4e^{t-1}dt, t^5e^{t-1}dt \rangle = \frac{2t^5}{5}e^{t-1} + \frac{3t^5}{5}e^{t-1} + \frac{t^6}{6}e^{t-1} \Big|_1^2$

$= e^{t-1}\left(\frac{5t^5}{5} + \frac{t^6}{6}\right)\Big|_1^2 = e^{t-1}\left(\frac{6t^5 + t^6}{6}\right)\Big|_1^2 = e^1\left(\frac{192}{6}\right) - \cancel{e^0}\left(\frac{6}{6}\right) = 32e - 1$

9) $F = y^2 i + (2xy + e^z) j + ye^z k$

$\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ y^2 & 2xy+e^z & ye^z \end{vmatrix} = i\left(\frac{d}{dy}ye^z - \frac{d}{dz}2xy + e^z\right) - j\left(\frac{d}{dx}ye^z - \frac{d}{dz}y^2\right) + k\left(\frac{d}{dx}2xy + e^z - \frac{d}{dy}y^2\right)$

$= i(e^z - e^z) - j(0 - 0) + k(2y - 2y) = \langle 0, 0, 0 \rangle$ conservative

$\frac{d}{dy}(xy^2 + g(y,z)) = 2xy + e^z + g'(x,z)$

$2xy = 2xy + e^z + g'(x,z) \quad g'(x,z) = -e^z \quad g(x,z) = -e^z + h(z)$

$f(x,y,z) = ye^z + xy^2$

$f(x,y,z) = xy^2 - e^z + h(z)$

$\frac{d}{dz}(xy^2 - e^z + h(z)) = ye^z \rightarrow zxy^2 - e^z + h'(z) = ye^z \rightarrow h'(z) = ye^z + e^z - zxy^2$

$h(z) = ye^z + e^z - xy^2$

13) $F = \langle z\sec^2 x, z, y+\tan x \rangle$

$$\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ z\sec^2 x & z & y+\tan x \end{vmatrix} = i\left(\tfrac{d}{dy} y+\tan x - \tfrac{d}{dz} z\right) - j\left(\tfrac{d}{dx} y+\tan x - \tfrac{d}{dz} z\sec^2 x\right) + k\left(\tfrac{d}{dx} z - \tfrac{d}{dy} z\sec^2 x\right)$$

$= i(1-1) - j(\sec^2 x - \sec^2 x) + k(0-0) = \langle 0,0,0 \rangle$ conservative

$\int (z\sec^2 x) = z\tan x + g(y,z)$

$\frac{d}{dy}(z\tan x + g(y,z)) = z + g'(x,z)$

$0 = z + g'(x,z) \quad g'(x,z) = -z$

$g(x,z) = -\frac{z^2}{2} + h(z)$

$f(x,y,z) = z\tan x - \frac{z^2}{2} + h(z)$

$\frac{d}{dz}\left(z\tan x - \frac{z^2}{2} + h(z)\right) = y + \tan x$

$\tan x - z + h'(z) = y + \tan x$

$h'(z) = y + z$

$h(z) = yz + \frac{z^2}{2}$

$f(x,y,z) = z\tan x + yz$

15) $F = \langle 2xy+5, x^2-4z, -4y \rangle$

$$\begin{vmatrix} i & j & k \\ \frac{d}{dx} & \frac{d}{dy} & \frac{d}{dz} \\ 2xy+5 & x^2-4z & -4y \end{vmatrix} = i\left(\tfrac{d}{dy}(-4y) - \tfrac{d}{dz}(x^2-4z)\right) - j\left(\tfrac{d}{dx}(-4y) - \tfrac{d}{dz}(2xy+5)\right) + k\left(\tfrac{d}{dx}(x^2-4z) - \tfrac{d}{dy}(2xy+5)\right)$$

$= i(-4+4) - j(0-0) + k(2x-2x) = \langle 0,0,0 \rangle$ conservative

$\int (2xy+5) = x^2y + 5x + g(y,z)$

$\frac{d}{dy}(x^2y + 5x + g(y,z)) = x^2 - 4z + g'(x,z)$

$x^2 = x^2 - 4z + g'(x,z)$

$g'(x,z) = 4z \quad g(x,z) = 2z^2$

$f(x,y,z) = x^2y + 5x + 2z^2 + h(z)$

$\frac{d}{dz}(x^2y + 5x + 2z^2 + h(z)) = -4y$

$4z + h'(z) = -4y$

$h'(z) = 4z - 4y \quad h(z) = 2z^2 - 4yz$

$f(x,y,z) = x^2y + 5x - 4yz$

17) $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$ over the path $r(t) = (t^2, \sin(\pi t/4), e^{t^2-2t})$ for $0 \le t \le 2$

$x = t^2 \quad y = \sin(\pi t/4) \quad z = e^{t^2-2t}$

$dx = 2t \quad dy = \cos(\pi t/4) \quad dz = (2t-2)e^{t^2-2t}$

$\int_0^2 2(t^2)(\sin(\pi t/4))(e^{t^2-2t})(2t) + (t^2)^2(e^{t^2-2t})(\cos(\pi t/4))$
$+ (t^2)(\sin(\pi t/4))((2t-2)e^{t^2-2t})$

$= \int_0^2 4t^3\sin(\frac{\pi t}{4})e^{t^2-2t} + t^4\cos(\frac{\pi t}{4})e^{t^2-2t} + (2t^3-2t^2)\sin(\frac{\pi t}{4})e^{t-2t}$

$= \int_0^2 e^{t^2-2t}\left(4t^3\sin(\frac{\pi t}{4}) + t^4\cos(\frac{\pi t}{4}) + (2t^3-2t^2)\sin(\frac{\pi t}{4})\right) \Rightarrow 16$

19) $f = x^2y - z$, $r_1 = \langle t, t, 0 \rangle$ for $0 \le t \le 1$ and $r_2 = \langle t, t^2, 0 \rangle$ for $0 \le t \le 1$

$x = t \quad y = t \quad z = 0$
$dx = 1 \quad dy = 1 \quad dz = 0$

$x = t \quad y = t^2 \quad z = 0$
$dx = 1 \quad dy = 2t \quad dz = 0$

$F = \langle 2xy, x^2, -1 \rangle$

$\int_0^1 \langle 2t^2, t^2, -1 \rangle \cdot \langle dt, dt, 0 \rangle$

$\int_0^1 \langle 2t^2dt, t^2dt, 0 \rangle = \frac{2}{3}t^3 + \frac{t^3}{3}\Big|_0^1 = \frac{2}{3} + \frac{1}{3} = 1$

$\int_0^1 \langle 2t^3, t^2, -1 \rangle \cdot \langle dt, 2tdt, 0 \rangle$

$= \int_0^1 \langle 2t^3dt, 2t^3dt, 0 \rangle = \frac{t^4}{2} + \frac{t^4}{2} + 0\Big|_0^1 = 1$