→ S1. Compute $\|r'(t)\|$
S2. Write out $f(r(t))$
S3. Compute line integral (Thm. on Scalar Line Integrals)

16.2, 16.3 HW

16.2: *3, 9, 11, 13, 17, 27, 29, 31, 35

3. $F = \langle y^2, x^2\rangle$; C is the curve $y = x^{-1}$ for $1 \le x \le 2$ (left to right)

(a) Calc. $F(r(t))$ & $dr = r'(t)dt$ for the parametrization of C given by $r(t) = \langle t, t^{-1}\rangle$

$F(r(t)) = \langle (t^{-1})^2, t^2\rangle = \langle t^{-2}, t^2\rangle$

$r'(t) = d/dt \langle t, t^{-1}\rangle = \langle 1, -t^{-2}\rangle \Rightarrow dr = \langle 1, -t^{-2}\rangle dt$

(b) Calc. dot product $F(r(t))\cdot r'(t)dt$ & evaluate $\int_C F\cdot dr$

$F(r(t))\cdot r'(t) = \langle t^{-2}, t^2\rangle \cdot \langle 1, -t^{-2}\rangle = t^{-2}\cdot 1 + t^2\cdot(-t^{-2}) = t^{-2} - 1$

$\int_C F\cdot dr = \int_1^2 F(r(t))\cdot r'(t)\,dt = \int_1^2 (t^{-2}-1)\,dt = -t^{-1} - t \Big|_1^2 = -\frac{1}{2}$

9. $f(x,y) = \sqrt{1+9xy}$, $y = x^3$ for $0 \le x \le 2$ (curve is parametrized by $r(t) = t, t^3$ for $0 \le t \le 2$)

S1. $r'(t) = d/dt\langle t, t^3\rangle = \langle 1, 3t^2\rangle \Rightarrow \|r'(t)\| = \sqrt{1+9t^4}$

S2. $f(r(t)) = \sqrt{1+9t\cdot t^3} = \sqrt{1+9t^4}$

S3. $\int_C f(x,y)\,ds = \int_0^2 f(r(t))\|r'(t)\|\,dt = \int_0^2 \sqrt{1+9t^4}\sqrt{1+9t^4} = \int_0^2 1+9t^4\,dt = \frac{298}{5} = 59.6$

11. $f(x,y,z) = z^2$, $r(t) = \langle 2t, 3t, 4t\rangle$ for $0 \le t \le 2$

S1. $r'(t) = d/dt\langle 2t, 3t, 4t\rangle = \langle 2,3,4\rangle \Rightarrow \|r'(t)\| = \sqrt{2^2+3^2+4^2} = \sqrt{29}$

S2. $f(r(t)) = 16t^2$

S3. $\int_C f(x,y,z)\,ds = \int_0^2 f(r(t))\|r'(t)\|\,dt = \int_0^2 16t^2\cdot\sqrt{29}\,dt = \sqrt{29}\cdot\frac{16}{3}t^3\Big|_0^2 = \frac{128\sqrt{29}}{3}$

13. $f(x,y,z) = xe^{z^2}$, piecewise linear path from $(0,0,1)$ to $(0,2,0)$ to $(1,1,1)$

$C_1: r_1(t) = (0, 2t, 1-t)$, $0 \le t \le 1$

$C_2: r_2(t) = (t, 2-t, t)$, $0 \le t \le 1$

$\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds$

Integral over C_1: $r_1'(t) = d/dt\langle 0, 2t, 1-t\rangle = \langle 0,2,-1\rangle \Rightarrow \|r'(t)\| = \sqrt{0+4+1} = \sqrt{5}$

$f(r_1(t)) = xe^{z^2} = 0\cdot e^{(1-t)^2} = 0$

Integral over C_2: $r_2'(t) = d/dt\langle t, 2-t, t\rangle = \langle 1,-1,1\rangle \Rightarrow \|r_2'(t)\| = \sqrt{1+1+1} = \sqrt{3}$

$f(r_2(t)) = xe^{z^2} = te^{t^2} \Rightarrow 1.488$

$\int_C f(x,y,z)\,ds \approx 1.488$

17. Calc. $\int_C 1\,ds$, where the curve C is parametrized by $r(t) = \langle 4t, -3t, 12t\rangle$ for $2 \le t \le 5$. What does this integral represent?

Compute $\|r'(t)\|$: $r'(t) = d/dt \langle 4t, -3t, 12t \rangle = \langle 4, -3, 12 \rangle$

$\Rightarrow \|r'(t)\| = \sqrt{4^2 + (-3)^2 + 12^2} = 13$

Compute Line Integral: $\int_C 1\,ds = \int_2^5 \|r'(t)\|\,dt = \int_2^5 13\,dt = 13(5-2) = 39$

This represents the dist. from the pt. $(8,-6,24)$ to the pt. $(20,-15,60)$.

27. $\int_C y\,dx - x\,dy$, parabola $y = x^2$ for $0 \le x \le 2$ (Parametrize C by $x(t) = t$, $y(t) = t^2$ for $0 \le t \le 2$. ($dx/dt = 1$ & $dy/dt = 2t$))

$$\int_C y\,dx - x\,dy = \int_0^2 (t^2\,dt - t(2t\,dt)) = \int_0^2 (-t^2)\,dt = -\frac{1}{3}t^3 \Big|_0^2 = -\frac{8}{3}$$

29. $\int_C (x-y)\,dx + (y-z)\,dy + z\,dz$, line seg. from $(0,0,0)$ to $(1,4,4)$

$(x(t) = t,\ y(t) = 4t,\ z(t) = 4t,\ 0 \le t \le 1) \Rightarrow (dx = dt,\ dy = 4dt,\ dz = 4dt)$

$$\int_C (x-y)\,dx + (y-z)\,dy + z\,dz = \int_0^1 ((t-4t)\cdot 1 + (4t-4t)\cdot 4 + 4t\cdot 4)\,dt$$

$$= \int_0^1 13t\,dt = \frac{13}{2}t^2 \Big|_0^1 = \frac{13}{2}$$

31. $\int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$, segment from $(1,0)$ to $(0,1)$

S1. Calc. Integrand: $r(t) = (1-t, t)$ $(0 \le t \le 1)$

$$F(r(t)) = \frac{1}{x^2+y^2}\langle -y, x \rangle = \frac{1}{(1-t)^2+t^2}\langle -t, 1-t \rangle$$

$$r'(t) = \langle -1, 1 \rangle$$

$$F(r(t))\cdot r'(t) = \frac{1}{(1-t)^2+t^2}\langle -t, 1-t \rangle \cdot \langle -1, 1 \rangle = \frac{t+1-t}{(1-t)^2+t^2} = \frac{1}{2t^2-2t+1}$$

S2. $$\int_C \frac{-y\,dx + x\,dy}{x^2+y^2} = \int_0^1 \frac{dt}{2t^2-2t+1} = \frac{1}{2}\int_0^1 \frac{dt}{t - \frac{1}{2}^2 + \frac{1}{4}} = \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{4}(\tan^2\theta+1)} = \int_{-\pi/4}^{\pi/4} d\theta = \frac{\pi}{2}$$

35. C_1: $r_1(t) = (0,0,t)$ $0 \le t \le 1$ $\quad r_1'(t) = \langle 0,0,1 \rangle$ $\quad (C = C_1 + C_2 + C_3)$

C_2: $r_2(t) = (0,t,1)$ $0 \le t \le 1$ $\Rightarrow$ $r_2'(t) = \langle 0,1,0 \rangle$

C_3: $r_3(t) = (t,1,1)$ $0 \le t \le 1$ $\quad r_3'(t) = \langle 1,0,0 \rangle$

$\int_{C_1} F\cdot dr = \int_0^1 F(r_1(t))\cdot r_1'(t)\,dt = \int_0^1 \langle e^t, e^{0-0}, e^0 \rangle \cdot \langle 0,0,1 \rangle\,dt = 1$

$\int_{C_2} F\cdot dr = \int_0^1 F(r_2(t))\cdot r_2'(t)\,dt = \int_0^1 \langle e^1, e^{0-t}, e^t \rangle \cdot \langle 0,1,0 \rangle\,dt = 1 - e^{-1}$

$\int_{C_3} F\cdot dr = \int_0^1 F(r_3(t))\cdot r_3'(t)\,dt = \int_0^1 \langle e^1, e^{t-1}, e^1 \rangle \cdot \langle 1,0,0 \rangle\,dt = e$

$\int_C F\cdot dr = 1 + (1-e^{-1}) + e = e + 2 - e^{-1}$

16.3: #1, 3, 5, 9, 13, 15, 17, 19 11/12/20

1. Let $f(x,y,z) = xy\sin(yz)$ & $F = \nabla f$. Evaluate $\int_C F \cdot dr$, where C is any path from $(0,0,0)$ to $(1,1,\pi)$.
$\int_C \nabla f \cdot dr = f(1,1,\pi) - f(0,0,0) = 1 \cdot 1 \sin\pi - 0 = 0$

3. $F(x,y) = \langle 3, 6y \rangle$, $f(x,y) = 3x + 3y^2$; $r(t) = \langle t, 2t^{-1} \rangle$ on interval $1 \le t \le 4$
$\nabla f = \langle df/dx, df/dy \rangle = \langle 3, 6y \rangle = F$
$\int_C F \cdot dr = f(r(4)) - f(r(1)) = f(4, 1/2) - f(1,2) = 3 \cdot 4 + 3 \cdot 1/4 - (3 \cdot 1 + 3 \cdot 4) = -9/4$

5. $F(x,y,z) = ye^z i + xe^z j + xye^z K$, $f(x,y,z) = xye^z$; $r(t) = \langle t^2, t^3, t-1 \rangle$ for $1 \le t \le 2$
$\nabla f = df/dx\, i + df/dy\, j + df/dz\, K = ye^z i + xe^z j + xye^z K = F$
$r(2) = (4,8,1)$; $r(1) = (1,1,0)$ $\Rightarrow \int_C F \cdot dr = f(4,8,1) - f(1,1,0) = 32e - 1$

9. $F = y^2 i + (2xy + e^z) j + ye^z K$
$dF_1/dy = d/dy\ y^2 = 2y$
$dF_2/dx = d/dx\ (2xy + e^z) = 2y$ $\Rightarrow \dfrac{dF_1}{dy} = \dfrac{dF_2}{dx}$
$dF_2/dz = d/dz\ (2xy + e^z) = e^z$
$dF_3/dy = d/dy\ (ye^z) = e^z$ $\Rightarrow \dfrac{dF_2}{dz} = \dfrac{dF_3}{dy}$
$dF_3/dx = d/dx\ (ye^z) = 0$
$dF_1/dz = d/dz\ (y^2) = 0$ $\Rightarrow \dfrac{dF_3}{dx} = \dfrac{dF_1}{dz}$
$f(x,y,z) = y^2 dx = xy^2 + f(y,z)$
$f(x,y,z) = (2xy + e^z) dy = xy^2 + ye^z + g(x,z)$
$f(x,y,z) = ye^z dz = ye^z + h(x,y)$
$xy^2 + f(y,z) = xy^2 + ye^z + g(x,z) = ye^z + h(x,y) \Rightarrow f(x,y,z) = xy^2 + ye^z + C$

13. $F = \langle z\sec^2 x, z, y + \tan x \rangle$
$dF_1/dy = d/dy\ (z\sec^2 x) = 0$
$dF_2/dx = d/dx\ (z) = 0$ $\Rightarrow \dfrac{dF_1}{dy} = \dfrac{dF_2}{dx}$
$dF_2/dz = d/dz\ (z) = 1$
$dF_3/dy = d/dy\ (y + \tan x) = 1$ $\Rightarrow \dfrac{dF_2}{dz} = \dfrac{dF_3}{dy}$
$dF_3/dx = d/dx\ (y + \tan x) = \sec^2 x$
$dF_1/dz = d/dz\ (z\sec^2 x) = \sec^2 x$ $\Rightarrow \dfrac{dF_3}{dx} = \dfrac{dF_1}{dz}$
$f(x,y,z) = \int z\sec^2 x\, dx = z\tan x + f(y,z)$
$f(x,y,z) = \int z\, dy = yz + g(x,z)$
$f(x,y,z) = \int (y + \tan x)\, dz = yz + z\tan x + h(x,y)$
$z\tan x + f(y,z) = yz + g(x,z) = yz + z\tan x + h(x,y)$
$f(x,y,z) = yz + z\tan x + C$

15. $F = \langle 2xy+5,\ x^2-4z,\ -4y \rangle$

$dF_1/dy = d/dy\,(2xy+5) = 2x$ $\Rightarrow \frac{dF_1}{dy} = \frac{dF_2}{dx}$

$dF_2/dx = d/dx\,(x^2-4z) = 2x$

$dF_2/dz = d/dz\,(x^2-4z) = -4$ $\Rightarrow \frac{dF_2}{dz} = \frac{dF_3}{dy}$

$dF_3/dy = d/dy\,(-4y) = -4$

$dF_3/dx = d/dx\,(-4y) = 0$ $\Rightarrow \frac{dF_3}{dx} = \frac{dF_1}{dz}$

$dF_1/dz = d/dz\,(2xy+5) = 0$

$f(x,y,z) = x^2y + 5x - 4yz + C$

☆. Evaluate $\int_C 2xyz\,dx + x^2z\,dy + x^2y\,dz$ over the path $r(t) = (t^2, \sin(\pi t/4), e^{t^2-2t})$ for $0 \le t \le 2$

$f(x,y,z) = x^2yz$; path begins at $r(0) = (0,0,1)$ & ends at $r(2) = (4,1,1)$

$f(4,1,1) - f(0,0,1) = 16 - 16 = 16$

19. $f = x^2y - z$, $r_1 = \langle t,t,0 \rangle$ for $0 \le t \le 1$, & $r_2 = \langle t, t^2, 0 \rangle$ for $0 \le t \le 1$

$F = \nabla f = \langle df/dx,\ df/dy,\ df/dz \rangle = \langle 2xy,\ x^2,\ -1 \rangle$

$r_1'(t) = \langle 1,1,0 \rangle$, $F(r_1(t)) = \langle 2t^2, t^2, -1 \rangle$

$r_2'(t) = \langle 1, 2t, 0 \rangle$, $F(r_2(t)) = \langle 2t^3, t^2, -1 \rangle$

$$\left.\begin{aligned} \int_0^1 F(r_1(t)) \cdot r_1'(t)\,dt &= \int_0^1 3t^2\,dt = t^3 \Big|_0^1 = 1 \\ \int_0^1 F(r_2(t)) \cdot r_2'(t)\,dt &= \int_0^1 4t^3\,dt = t^4 \Big|_0^1 = 1 \end{aligned}\right\} =$$