No.

Date

Exercise 15.3

Q3. $f(x,y,z) = xe^{y-2z}$; $0 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 1$

$$\int_0^2 \int_0^1 \int_0^1 xe^{y-2z}\, dz\, dy\, dx$$

$$\Rightarrow \int_0^1 xe^{y-2z}\, dz$$
$$= x \int_0^1 e^{y-2z}\, dz$$
$$= x \cdot -\tfrac{1}{2} e^{y-2z} \Big|_0^1$$
$$= x \cdot \left(-\tfrac{1}{2} e^{y-2} + \tfrac{1}{2} e^{y}\right)$$
$$= -\tfrac{1}{2} x \cdot e^{y-2} + \tfrac{1}{2} x e^{y}$$

$$\int_0^1 -\tfrac{1}{2} x \cdot e^{y-2} + \tfrac{1}{2} x e^{y}\, dy$$
$$= \tfrac{1}{2} x \int_0^1 (-e^{y-2} + e^{y})\, dy$$
$$= \tfrac{1}{2} x \cdot \quad -e^{y-2} + e^{y} \Big|_0^1$$
$$= \tfrac{1}{2} x \cdot (-e^{-1} + e^{1} + e^{-2} - e^{0})$$
$$= \tfrac{1}{2} x \cdot (-e^{-1} + e^{1} + e^{-2} - 1)$$

$$\int_0^2 \tfrac{1}{2} x(-e^{-1} + e^{1} + e^{-2} - 1)\, dx$$
$$= (-e^{-1} + e^{1} + e^{-2} - 1) \tfrac{1}{4} x^2 \Big|_0^2$$
$$= (-e^{-1} + e^{1} + e^{-2} - 1) \cdot 1$$
$$= -e^{-1} + e^{1} + e^{-2} - 1$$

Q5. $f(x,y,z) = (x-y)(y-z)$ $[0,1] \times [0,3] \times [0,3]$

$$\int_0^1 \int_0^3 \int_0^3 (x-y)(y-z)\, dz\, dy\, dx$$

$$\Rightarrow \int_0^3 xy - xz - y^2 + yz\, dz$$
$$= xyz - \tfrac{1}{2} x z^2 - y^2 z + \tfrac{1}{2} y z^2 \Big|_0^3$$
$$= 3xy - \tfrac{9}{2} x - 3y^2 + \tfrac{9}{2} y - 0$$

$$\Rightarrow \int_0^3 3xy - \tfrac{9}{2} x - 3y^2 + \tfrac{9}{2} y\, dy$$
$$= \tfrac{3}{2} x y^2 - \tfrac{9}{2} x y - y^3 + \tfrac{9}{4} y^2 \Big|_0^3$$
$$= \tfrac{27}{2} x - \tfrac{27}{2} x - 27 + \tfrac{9}{4} \cdot 9$$
$$= \frac{27}{-4}$$

No.

Date

$$\int_0^1 -\frac{27}{4}\,dx$$
$$= -\frac{27}{4} x \Big|_0^1$$
$$= -\frac{27}{4}$$

Q7. $f(x,y,z) = (x+z)^3$; $[0,a]\times[0,b]\times[0,c]$

$$\int_0^a\int_0^b\int_0^c (x+z)^3\,dz\,dy\,dx$$

$$\Rightarrow \int_0^c (x+z)^3\,dz$$
$$= \frac{1}{4}(x+z)^4 \Big|_0^c$$
$$= \frac{1}{4}(x+c)^4 - \frac{x^4}{4}$$
$$= \frac{(x+c)^4 - x^4}{4}$$

$$\int_0^b \frac{(x+c)^4 - x^4}{4}\,dy$$
$$= \frac{1}{4}\int_0^b (x+c)^4 - x^4\,dy$$
$$= \frac{1}{4}\cdot\left[(x+c)^4\cdot y - x^4 y \Big|_0^b\right]$$
$$= \frac{1}{4}\cdot\left((x+c)^4\cdot b - x^4 b\right)$$
$$= \frac{1}{4}b\left[(x+c)^4 - x^4\right]$$

$$\int_0^a \frac{1}{4}b[(x+c)^4 - x^4]\,dx$$
$$= \frac{1}{4}b\left(\frac{1}{5}(x+c)^5 - \frac{1}{5}x^5\right)\Big|_0^a$$
$$= \frac{1}{4}b\cdot\left(\frac{(a+c)^5}{5} - \frac{a^5}{5} - \frac{c^5}{5}\right)$$
$$= \frac{1}{20}b\cdot\left((a+c)^5 - a^5 - c^5\right)$$

Q9. $f(x,y,z) = x+y$; $W = y\le z\le x,\ 0\le y\le x,\ 0\le x\le 1$

$$\int_0^1\int_0^x\int_y^x (x+y)\,dz\,dy\,dx$$

$$\Rightarrow \int_y^x (x+y)\,dz$$
$$= (x+y)z\Big|_y^x$$

Campus $= (x+y)\cdot x - (x+y)\cdot y = (x+y)\cdot(x-y)$

No.
Date

$$\int_0^x (x+y)(x-y)\,dy$$
$$=\int_0^x x^2-y^2 \cancel{-2xy}\,dy$$
$$= x^2y-\frac{1}{3}y^3 \cancel{-xy^2} \Big|_0^x$$
$$= x^3-\frac{1}{3}x^3 \cancel{-x^3}$$
$$= -\frac{1}{3}x^3+x^3$$

$$\int_0^1 x^3-\frac{1}{3}x^3\,dx$$
$$=\frac{x^4}{4}-\frac{1}{12}x^4 \Big|_0^1$$
$$= -\frac{1}{12}+\frac{1}{4}$$
$$=\frac{1}{6}$$

Q11. $f(x,y,z)=xyz$; $W: 0\le z\le 1,\ 0\le y\le \sqrt{1-x^2},$
$0\le x\le 1$

$$\int_0^1\int_0^1\int_0^{\sqrt{1-x^2}} xyz\,dy\,dz\,dx$$

$$\Rightarrow \int_0^{\sqrt{1-x^2}} xyz\,dy$$
$$= xz\cdot \frac{1}{2}y^2\Big|_0^{\sqrt{1-x^2}}$$
$$= xz\cdot\frac{1}{2}(1-x^2)$$
$$= \frac{(1-x^2)xz}{2}$$

$$\int_0^1 \frac{(1-x^2)xz}{2}\,dz$$
$$=\frac{1}{2}(1-x^2)x\int_0^1 z\,dz$$
$$=\frac{(1-x^2)x}{2}\cdot\frac{1}{2}z^2\Big|_0^1$$
$$=\frac{(1-x^2)x}{4}$$

$$\int_0^1 \frac{(1-x^2)x}{4}\,dx$$
$$=\frac{1}{4}\int_0^1 (1-x^2)x\,dx$$
$$=\frac{1}{4}\cdot\left(\frac{1}{2}x^2-\frac{1}{4}x^4\right)\Big|_0^1$$
$$=\frac{1}{4}\cdot\left(\frac{1}{2}-\frac{1}{4}\right)=\frac{1}{16}$$

No.

Date

Q13. $f(x,y,z) = e^{z}$ $\quad W: x+y+z \leq 1 \quad x \geq 0 \quad y \geq 0 \quad z \geq 0$

when x, y is mini. $z = 1 \quad \therefore 0 \leq z \leq 1$

when $z = 0$, $\quad x+y = 1 \quad y = 1-x \quad 0 \leq y \leq 1-x$

$$\therefore \int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^{z} \, dz \, dy \, dx$$

$$\Rightarrow \int_0^{1-x-y} e^{z} \, dz$$

$$= e^{z} \Big|_0^{1-x-y}$$

$$= e^{1-x-y} - 1$$

$$\int_0^{1-x} e^{1-x-y} - 1 \, dy$$

$$= -e^{1-x-y} - y \Big|_0^{1-x}$$

$$= -e^{1-x-1+x} - 1 + x + e^{1-x}$$

$$= -e^{0} - 1 + x + e^{1-x}$$

$$= -2 + x + e^{1-x}$$

$$\int_0^1 -2 + x + e^{1-x} \, dx$$

$$= -2x + \frac{1}{2}x^2 + (-e^{1-x}) \Big|_0^1$$

$$= -2 + \frac{1}{2} - e^{0} + e^{1}$$

$$= -3 + \frac{1}{2} + e^{1}$$

$$= e - \frac{5}{2}$$

No.

Date

Q15. $y=0 \rightarrow xz$ plane.

$z=\sqrt{9-x^2-y^2}$

$0 \leq z \leq \sqrt{9-x^2-y^2}$

$D=\{(x,y,z): \text{tbd}1 \leq x \leq \text{tbd}2,\ \text{tbd}3 \leq y \leq \text{tbd}4$

$0 \leq z \leq \sqrt{9-x^2-y^2}$

type I: $0 \leq x \leq 1 \quad 0 \leq y \leq x$

$\therefore D=\{(x,y,z): 0 \leq x \leq 1 \quad 0 \leq y \leq x,\ 0 \leq z \leq \sqrt{9-x^2-y^2}$

$\therefore D=\{(x,y,z): y \leq x \leq 1 \quad 0 \leq y \leq 1,\ 0 \leq z \leq \sqrt{9-x^2-y^2}$

$$\int_0^1 \int_0^x \int_0^{\sqrt{9-x^2-y^2}} z \, dz \, dy \, dx$$

$= \frac{25}{12}$

No.

Date

Q17. $f(x,y,z) = x \quad x \geq 0 \; y \geq 0 \; z \geq 0 \quad z = y^2 \quad z = 8 - 2x^2 - y^2$

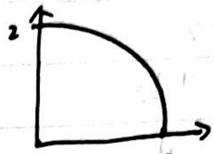
Ans: $y^2 = 8 - 2x^2 - y^2$

$2y^2 = 8 - 2x^2$

$y^2 = 4 - x^2$

$x^2 + y^2 = 4$

$(x^2 + y^2 = 4. \; x = 0..2 \quad y = 0..2)$



$y^2 \leq z \leq 8 - 2x^2 - y^2$

$\int_0^2 \int_0^{\sqrt{4-x^2}} x(8 - 2x^2 - 2y^2)\,dy\,dx$

$= \frac{128}{15}$

$\int_{y^2}^{8-2x^2-y^2} x \; dz$

$= x(8 - 2x^2 - y^2 - y^2)$

$= x \cdot (8 - 2x^2 - 2y^2)$

other way:

$x = r\cos(t) \quad y = r\sin(t) \qquad dA = dr\,dt \cdot r$

$x^2 + y^2 = r^2$

$\iint r\cos(t) \cdot (8 - 2(r\cos t)^2 - 2(r\sin t)^2)$

$= \int_0^{\frac{\pi}{2}} \int_0^2 r\cos t \cdot (8 - 2r^2)\,r, \; d\cancel{x}^{\,r}\,dt$

$= \int_0^2 r^2(8 - 2r^2)\,dr \quad \int_0^{\frac{\pi}{2}} \cos t \; dt.$

$= \frac{8}{3}r^3 - \frac{2}{5}r^5 \Big|_0^2 \quad \cancel{4r^2 - \frac{r^4}{2} \Big|_0^2}$

$= \sin t \Big|_0^{\frac{\pi}{2}}$

$= \cancel{16 - \frac{16}{2} - 0}$

$= 1 - 0$

$= 1$

$= \cancel{8} \; \frac{64}{3} - \frac{64}{5}$

$= \frac{128}{15}$

$\frac{128}{15} \times 1 = \frac{128}{15}$

KOKUYO

Exercise 15.4

Q1. $f(x,y)=\sqrt{x^2+y^2} \quad x^2+y^2 \le 2.$

$\because x^2+y^2=r^2 \qquad y^2=2-x^2$

$r^2 \le 2 \qquad y=\sqrt{2-x^2}$

$r^2=2 \qquad x=\sqrt{2-y^2}.$

$r=\pm\sqrt{2}$

$\therefore r=\sqrt{2} \qquad \therefore 0 \le r \le \sqrt{2} \quad 0 \le \theta \le 2\pi$

$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2} \cdot r \, dr d\theta$

$= \frac{4\sqrt{2}\pi}{3}$

Q5. $f(x,y)=y(x^2+y^2)^{-1} \quad y \ge \frac{1}{2}, \quad x^2+y^2 \le 1.$

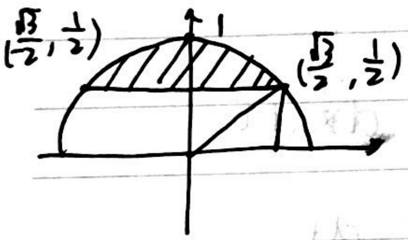
$x^2+y^2 \le 1 \qquad y=\sqrt{1-x^2}$

$r^2 \le 1 \qquad \frac{1}{2}=\sqrt{1-x^2}$

$r=\pm 1 \qquad \frac{1}{4}=1-x^2$

$\therefore 0 \le r \le 1 \qquad x^2=\frac{3}{4} \qquad \therefore (\frac{\sqrt{3}}{2},\frac{1}{2})$

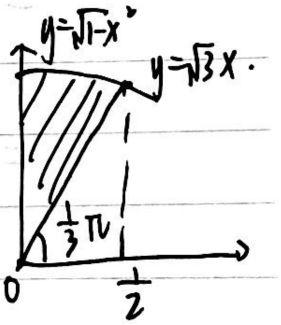
$x=\pm\sqrt{\frac{3}{4}}=\pm\frac{\sqrt{3}}{2} \qquad (-\frac{\sqrt{3}}{2},\frac{1}{2})$



$\therefore \quad \le r$

Q9. $\int_0^{\frac{1}{2}}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x \, dy dx \qquad \tan^{-1}\left(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right)=\frac{1}{3}\pi$

$0 \le x \le \frac{1}{2}$



$y=\sqrt{1-x^2} \qquad \sqrt{3}x=\sqrt{1-x^2}$

$y=\sqrt{3}x \qquad 3x^2=1-x^2 \qquad \int_{\frac{1}{3}\pi}^{\frac{1}{2}\pi}\int_0^1 r^2\cos\theta \, dr d\theta$

$x=\sqrt{1-y^2} \qquad 4x^2=1 \qquad =\int_{\frac{\pi}{3}}^{\frac{\pi}{2}}(\cos\theta\int_0^1 r^2 dr)d\theta$

$0 \le r \le 1 \qquad x^2=\frac{1}{4}$

$\therefore \cancel{0 \le r \le \frac{1}{2}} \qquad x=\pm\frac{1}{2} \qquad = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3}\cos\theta \, d\theta.$

$\because x \ge 0 \therefore x=\frac{1}{2}$

No.

Date

$= \frac{2-\sqrt{3}}{3} \approx 0.045.$

Q19. $f(x,y) = x-y, \quad x^2+y^2 \leq 1 \qquad x+y \geq 1$

$x^2+y^2 = r^2 \qquad r^2 \leq 1 \qquad x+y \geq 1$

$y = \sqrt{1-x^2} \qquad r = 1 \qquad y \geq 1-x$

$\therefore 0 \leq r \leq 1$

$\sqrt{1-x^2} = 1-x$

$1-x^2 = 1+x^2-2x$

$2x-x^2=0$

$x(2-x)=0$

$x=0 \text{ or } 2.$

$\int_0^{\frac{\pi}{2}} \int_0^1 r^2(\cos\theta - \sin\theta)\, dr\, d\theta$

$= \int_0^{\frac{\pi}{2}} \frac{1}{3}(\cos\theta - \sin\theta\ d\theta$

$= \frac{1}{3} \cdot (1-1)$

$= 0$

Q27. $f(x,y,z) = x^2+y^2 \ ; \ x^2+y^2 \leq 9 \quad 0 \leq z \leq 5$

$r^2 \leq 9$

$r = \pm 3$

$0 \leq r \leq 3$

$y^2 = 9-x^2$

$y = \sqrt{9-x^2}$

$\therefore \int_0^{2\pi} \int_0^3 \int_0^5 r^3\, dz\, dr\, d\theta$

$= \int_0^{2\pi} \int_0^3 r^3 \cdot 5\, dr\, d\theta$

$= \int_0^{2\pi} \frac{405}{4}\, d\theta$

$= \frac{405}{4} \cdot 2\pi = \frac{405\pi}{2}$

KOKUYO

Q31. $f(x,y,z) = z,\quad x^2+y^2 \le z \le 9$

$z = x^2+y^2 \qquad\qquad x^2+y^2 \le 9$

$z = r^2 \qquad\qquad r^2 \le 9$

$r^2 \le z \le 9 \qquad\qquad r = 3$

$\therefore 0 \le r \le 3$

$$\int_0^{2\pi}\int_0^3\int_{r^2}^9 z\cdot r \, dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left(r\cdot \frac{z^2}{2}\Big|_{r^2}^{9} \right) dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^3 \left(\frac{81}{2}r - \frac{r^5}{2} \right) dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{81r^2}{4} - \frac{r^6}{12} \right)\Big|_0^3 d\theta$$

$$= \int_0^{2\pi} \left(\frac{3^2\cdot 3^4}{4} - \frac{3^6}{12} \right) d\theta$$

$$= \int_0^{2\pi} 3^6\cdot\left(\frac{1}{4} - \frac{1}{12} \right) d\theta$$

$$= \cancel{2\pi}\ \frac{3^6}{6}\int_0^{2\pi} 1\, d\theta$$

$$= \frac{3^6}{6}\cdot 2\pi$$

$$= 243\pi$$

Q47. $f(x,y,z) = x^2+y^2;\quad \rho \le 1$

$\rho = R$

$\therefore R \le 1$

$0 \le R \le 1$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} \left(\rho^2 - (\rho\cos\phi)^2\right)\cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

No.

Date

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1}\left(\rho^2-(\rho\cos\phi)^2\right)\cdot\rho^2\sin\phi \, d\rho \, d\theta \, d\phi$$

$$=\int_0^{\pi}\int_0^{2\pi}\int_0^{1}\rho^4\cdot\left(\sin\phi-\sin\phi\cos^2\phi\right)d\rho \, d\theta \, d\phi$$

$$=\int_0^{\pi}\cdot 2\pi\cdot\frac{1}{5}\left(\sin\phi-\sin\phi\cos^2\phi\right) \, d\phi$$

$$=\cancel{\int_0^{\pi}} \quad \frac{2\pi}{5}\cdot\frac{4}{3}$$

$$=\frac{8}{15}\pi.$$

No.

Date . .

Q51. $f(x,y,z) = z \quad 0 \le \theta \le \frac{\pi}{3},\ 0 \le \phi \le \frac{\pi}{2},\ 1 \le \rho \le 2.$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_1^2 \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \left(\cos\phi \sin\phi \ \frac{\rho^4}{4} \Big|_1^2 \right) d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \left(4\cos\phi\sin\phi - \frac{1}{4}\cos\phi\sin\phi \right) d\theta \, d\phi$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \left(2\sin 2\phi - \frac{1}{8}\sin 2\phi \right) d\theta \, d\phi.$$

$$= \frac{\pi}{3} \int_0^{\frac{\pi}{2}} \left(2\sin 2\phi - \frac{1}{8}\sin 2\phi \right) d\phi$$

$$= \frac{\pi}{3} \cdot \left(-1 \cdot \cos 2\phi + \frac{1}{16}\cos 2\phi \right) \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{3} \cdot \left(1 + (-\frac{1}{16}) + 1 - \frac{1}{16} \right)$$

$$= \frac{5}{8}\pi.$$