15.3-15.4 (Nov. 1st)

15.3: # 3, 5, 7, 9, 11, 13, 15, 17

15.4: # 1, 5, 9, 19, 27, 31, 47, 51

15.3: # 3, 5, 7, 9, 11, 13, 15, 17

3) $\int_0^2\int_0^1\int_0^1 xe^{y-2z}\,dz\,dy\,dx$

$= \int_0^2\int_0^1 \left[-\frac{xe^{y-2z}}{2}\right]_0^1 dy\,dx$

$= \int_0^2\int_0^1 \left[\frac{xe^{y-2(0)}}{2} - \frac{xe^{y-2(1)}}{2}\right] dy\,dx$

$= \int_0^2\int_0^1 \left[\frac{xe^{y}}{2} - \frac{xe^{y-2}}{2}\right]_0^1 dx$

$= \int_0^2 \left[\frac{xe^{1}}{2} - \frac{xe^{1-2}}{2}\right] dx$

$= \frac{1}{2}\left[\frac{ex^2}{2} - \frac{x^2}{2}e^{-1}\right]_0^2$

$= \frac{1}{4}(4e - 4e^{-1}) = \boxed{e - \frac{1}{e}}$

5) $\int_0^3\int_0^3\int_0^1 (xy - xz - y^2 + yz)\,dx\,dy\,dz$

$= \int_0^3\int_0^3 \left[\frac{x^2y}{2} - \frac{x^2z}{2} - xy^2 + xyz\right]_0^1 dy\,dz$

$= \int_0^3\int_0^3 \left(\frac{y}{2} - \frac{z}{2} - y^2 + yz\right) dy\,dz$

$= \int_0^3 \left[\frac{y^2}{4} - \frac{zy}{2} - \frac{y^3}{3} + \frac{zy^2}{2}\right]_0^3 dz$

$= \int_0^3 \left(3z - \frac{27}{4}\right) dz$

$= \boxed{-\frac{27}{4}}$

7) $\int_0^c\int_0^b\int_0^a (x+z)^3\,dx\,dy\,dz$

$\int_0^c\int_0^b \left[\frac{(x+z)^4}{4}\right]_0^a dy\,dz$

$\int_0^c\int_0^b \left(\frac{(a+z)^4}{4} - \frac{(0+z)^4}{4}\right) dy\,dz$

$= \int_0^c \left[\frac{(a+z)^4 y}{4} - \frac{z^4 y}{4}\right] dz$

$= \left[\frac{(a+z)^5 b}{20} - \frac{z^5 b}{20}\right]_0^c$

$= \boxed{\frac{(a+c)^5 b - c^5 b - a^5 b}{20}}$

9) $\int_0^1\int_0^x\int_y^x (x+y)\,dz\,dy\,dx$

$= \int_0^1\int_0^x (x+y)z\Big|_y^x\,dy\,dx$

$= \int_0^1\int_0^x (x^2 - y^2)\,dy\,dx$

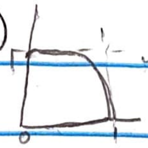
$= \int_0^1 \left[x^2y - \frac{y^3}{3}\right]_0^x dx$

$= \int_0^1 \frac{2}{3}x^3\,dx$

$= \left[\frac{1}{6}x^4\right]_0^1$

$= \boxed{1/6}$

11)



$\iiint_E xyz\,dV = \iint_D \left(\int_0^1 xyz\,dz\right) dA$

$= \iint_D \frac{xyz^2}{2}\Big|_0^1 dA = \iint_D \frac{xy}{2}\,dA$

$= \int_0^1 \frac{xy^2}{4}\Big|_{y=0}^{y=\sqrt{1-x^2}} = \frac{x^2}{8} - \frac{x^4}{16}\Big|_0^1$

$= \boxed{\frac{1}{16}}$

13) $x+y+z=1$, $z=0 \to x+y=1$

$z=1-x-y$, $z=0$,

$\iiint_W e^z\,dV = \iint_D \left(\int_0^{1-x-y} e^z\,dz\right) dA$

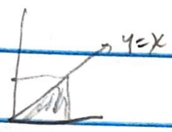
$= \iint_D (e^{1-x-y}-1)\,dA$

$= \int_0^1 \left(e^{1-x-y} - x \Big|_0^{1-y}\right) dy$

$= \int_0^1 (e^{1-y}+y-2)\,dy = -e^{1-y} + \frac{y^2}{2} - 2y \Big|_{y=0}^{1}$

$= -1+\frac{1}{2}-2-(-e^1) = \boxed{e-\frac{5}{2}}$

15)



$\iiint_V z\,dV = \iint_D \left(\int_0^{\sqrt{9-x-y^2}} z\,dz\right) dA$

$= \iint_D \frac{9-x-y^2}{2}\,dA$

$\int_0^1\int_0^x \frac{9-x-y^2}{2}\,dy\,dx = \int_0^1 \frac{9y-x^2y-y^3/3}{2}\Big|_{y=0}^{y=1} dx$

$= \frac{9x^2}{4} - \frac{x^4}{6}\Big|_0^1 = \frac{9}{4}-\frac{1}{6} = \boxed{\frac{25}{12}}$

17) $y^2 = 8-2x^2-y^2$

$x^2+y^2=4$

$x=\pm 2$, $y=\pm\sqrt{4-x^2}$

$\iiint_E f(x,y,z)\,dV = \int_{x=0}^{2}\int_{y=0}^{\sqrt{4-y^2}}\int_{z=y^2}^{8-2x^2-y^2} x\,dz\,dy\,dx$

$\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} [x(8-2x^2-y^2)-x(y^2)]\,dy\,dx$

$\int_{x=0}^{2}\int_{y=0}^{\sqrt{4-x^2}} [8x-2x^3-2xy^2]\,dy\,dx$

$\int_{x=0}^{2} \left[\frac{4}{3}x\sqrt{4-x^2}(4-x^2)\right]dx$

$= \frac{4}{3}\int_{x=0}^{2} [x(4-x^2)^{3/2}]\,dx$

$4-x^2=t$

$x\,dx = -\frac{1}{2}dt$

$\frac{4}{3}\int_{t=4}^{0} (t)^{3/2}\left(-\frac{1}{2}dt\right)$

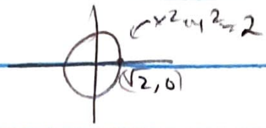
$= -\frac{2}{3}\left[\frac{t^{5/2}}{5/2}\right]_{t=4}^{0}$

$= -\frac{4}{15}(0-32)$

$= \boxed{\frac{128}{15}}$

15.4: #1, 5, 9, 19, 27, 31, 47, 51

1) $r = \sqrt{2}$



$f(x,y) = g(r,\theta) = r$

$r \in [0, \sqrt{2}]$; $\theta \in [0, 2\pi)$

$= \int_0^{2\pi} \int_0^{\sqrt{2}} g(r,\theta)\, dr\, d\theta$ $\left(\int_D \int dx\, dy = \int_0^{2\pi} \int_0^{\sqrt{2}} r\, dr\, d\theta \right.$

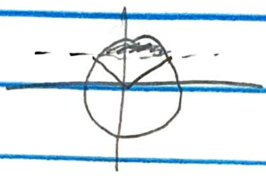
$= \int_0^{2\pi} \int_0^{\sqrt{2}} r \cdot dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{r^3}{3} \right]_0^{\sqrt{2}} d\theta$ $\left(\int r^2 dr = \frac{r^3}{3} \right)$

$= \int_0^{2\pi} (d\theta) \frac{2\sqrt{2}}{3} = \boxed{\frac{4\sqrt{2}\pi}{3}}$

5) $f(x,y) = y(x^2+y^2)^{-1}$

$\sin\alpha = \frac{1}{2}$, $\alpha = \frac{\pi}{6}$



$\theta \in [\frac{\pi}{6}, \frac{5\pi}{6}]$

$y = r\sin\theta$, $x^2+y^2 = r^2$

$\therefore f(x,y) = g(r,\theta) = \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$

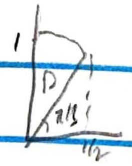
$= \int_D \int f(x,y)\, dx\, dy = \int_{\pi/6}^{5\pi/6} \int_{\cos\frac{\pi}{6}}^{1} \frac{\sin\theta}{r} r\, dr\, d\theta$

$= \int_{\pi/6}^{5\pi/6} \sin\theta\, d\theta - \int_{\pi/6}^{5\pi/6} \frac{d\theta}{2}$

$= [-\cos\theta]_{\pi/6}^{5\pi/6} - \frac{1}{2}\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$

$= \boxed{\sqrt{3} - \frac{\pi}{3}}$

9)



$0 \le x \le \frac{1}{2}$, $\sqrt{3}x \le y \le \sqrt{1-x^2}$

$\tan\theta = \sqrt{3}$, $\theta = \pi/3$

$\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\, dy\, dx = \int_{\pi/3}^{\pi/2} \int_0^1 (r\cos\theta) r\, dr\, d\theta$

$= \int_{\pi/3}^{\pi/2} \int_0^1 r^2 \cos\theta\, dr\, d\theta$

$= \int_{\pi/3}^{\pi/2} \left(\frac{r^3}{3} \cos\theta \right)_0^1 d\theta = \boxed{\frac{1}{3} - \frac{\sqrt{3}}{6}}$

19) $f(x,y) = x - y = r\cos\theta - r\sin\theta$

$$\iint_D (x-y)\,dA = \int_0^{\pi/2} \int_{\cos\theta+\sin\theta}^{1} r(\cos\theta - \sin\theta)\, r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_{\cos\theta+\sin\theta}^{1} r^2 (\cos\theta - \sin\theta)\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{r^3}{3}(\cos\theta - \sin\theta)\right]_{\cos\theta+\sin\theta}^{1} d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(\cos\theta+\sin\theta)^3}\,d\theta$$

$$\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(\cos\theta+\sin\theta)^3}\,d\theta = \int_1^1 \frac{du}{u^3} = 0$$

$$= \boxed{0}$$

27) $f(x,y,z) = x^2+y^2 \Rightarrow r^2$

$$\iiint_W f(x,y,z)\,dV = \iiint_W f(r,\theta,z)\, r\,dr\,d\theta\,dz$$

$$= \int_0^5 \int_0^{2\pi} \int_0^3 r^3\,dr\,d\theta\,dz$$

$$= \int_0^5 \int_0^{2\pi} \left[\frac{r^4}{4}\right]_0^3 d\theta\,dz$$

$$= \frac{81}{4}\int_0^5 \int_0^{2\pi} d\theta\,dz = \boxed{\frac{405\pi}{2}}$$

31) $f(x,y,z) = z$

$0 \le \theta \le 2\pi$

$$\iiint_W f(x,y,z)\,dV = \int_0^3 \int_0^{2\pi} \int_{r^2}^{9} z r\,dz\,d\theta\,dr$$

$$= \int_0^3 \int_{r^2}^{9} r z\,dz\,dr \cdot \int_0^{2\pi} d\theta$$

$$= 2\pi\left(\left(\int_0^2 \frac{81r}{2}\,dr\right) - \int_0^2 \frac{r^5}{2}\,dr\right)$$

$$= \pi\left(\frac{729}{2} - \frac{729}{6}\right) = \boxed{243\pi}$$

47) W. $0 \le \theta \le 2\pi$

$f(x,y,z) = x^2+y^2$

$= \rho^2\sin^2\theta\,(1) = \rho^2\sin\theta$

$$= \iiint_W (x^2+y^2)\,dV =$$

$$= \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^4 \sin^3\theta\,d\rho\,d\theta\,d\theta)$$

$$= 2\pi\left(\frac{-\sin^2\theta\cos\theta - 2\cos\theta}{3}\right)_0^{\pi} \left(\frac{\rho^5}{5}\right)_0^1$$

$$= \boxed{8\pi/15}$$

51) $f(x,y,z) = \rho\cos\phi$

$$\iiint_W z\,dV = \int_0^{\pi/3} \int_0^{\pi/2} \int_1^2 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi/3} \int_0^{\pi/2} \int_1^2 \rho^3 \cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \frac{\pi}{3}\left(-\frac{1}{4}\cos 2\phi\right)_0^{\pi/2} \left(\rho^4/4\right)_1^2 = \boxed{5\pi/8}$$