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15.3: 3, 5, 7, 9, 11, 13, 15, 17.

#3 $f(x,y,z) = xe^{y-2z}$; $0 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 1$

$$\int_0^2 \int_0^1 \int_0^1 xe^{y-2z} \, dz\,dy\,dx$$

$$\int_0^1 xe^{y-2z} \, dz = -\tfrac{1}{2} \cdot x \cdot e^y \cdot e^{-2z} \Big|_0^1 = -\frac{xe^y}{2} \cdot \left(e^{-2} - e^0\right) = -\frac{x}{2} \cdot \left(e^{y-2} - e^y\right)$$

$x \cdot e^y \cdot e^{-2z}$

$$-\frac{x}{2}\int_0^1 \left(e^{y-2} - e^y\right) dy = -\frac{x}{2}\left(e^{y-2} - e^y\right)\Big|_0^1 = -\frac{x}{2}\left[\left(e^{-1} - e^1\right) - \left(e^{-2} - 1\right)\right] = -\frac{e^{-1} - e - e^{-2} + 1}{2} \cdot x$$

$$-\frac{e^{-1} - e - e^{-2} + 1}{2}\int_0^2 x \, dx = -\frac{e^{-1} - e - e^{-2} + 1}{2}\left(\frac{x^2}{2}\right)\Big|_0^2 = -\frac{e^{-1} - e - e^{-2} + 1}{2} \cdot (2 - 0) = \boxed{-e^{-1} - e - e^{-2} + 1}$$

#5 $f(x,y,z) = (x-y)(y-z)$; $[0,1] \times [0,3] * [0,3]$

$$\int_0^1 \int_0^3 \int_0^3 (x-y)(y-z) \, dz\,dy\,dx$$

$$\int_0^3 (x-y)(y-z) \, dz = (x-y)\left(yz - \frac{z^2}{2}\right)\Big|_0^3 = (x-y)\left(3y - \frac{9}{2} - 0\right) = 3xy - \frac{9x}{2} - 3y^2 + \frac{9y}{2}$$

$$\int_0^3 3xy - \frac{9x}{2} - 3y^2 + \frac{9y}{2} \, dy = 3x\frac{y^2}{2} - \frac{9xy}{2} - y^3 + \frac{9y^2}{4}\Big|_0^3 = 3x\frac{9}{2} - \frac{27x}{2} - 27 + \frac{81}{4}$$

$$\int_0^1 3x\frac{9}{2} - \frac{27x}{2} - 27 + \frac{81}{4} \, dx = \frac{27}{4}x^2 - \frac{27x^2}{4} - 27x + \frac{81x}{4}\Big|_0^1 = \frac{27}{4} - \frac{27}{4} - 27 + \frac{81}{4} = \boxed{-\frac{27}{4}}$$

#7 $$\int_0^a \int_0^b \int_0^c (x+y-z)^2 \, dz\,dy\,dx$$

$$\int_0^c (x+y-z)^2 \, dz = -\int_{x+y}^{x+y-c} u^2 \, du = \frac{u^2}{2}\Big|_{x+y}^{x+y-c} = \frac{(x+y-c)^3}{3} - \frac{(x+y)^3}{3}$$

$u = x+y-z \quad du = -1dz$

$$\frac{1}{3}\int_0^b (x+y-c)^3 - (x+y)^3 \, dy = \frac{(x+y-c)^4}{12} - \frac{(x+y)^4}{12}\Big|_0^b = \left(\frac{(x+b-c)^4}{12} - \frac{(x+b)^4}{12}\right) - \left(\frac{(x-c)^4}{12} - \frac{x^4}{12}\right)$$

$$\frac{1}{3}\int_0^a \frac{(x+b-c)^4}{12} - \frac{(x+b)^4}{12} - \frac{(x-c)^4}{12} + \frac{x^4}{12} \, dx = \frac{(x+b-c)^5}{60} - \frac{(x+b)^5}{60} - \frac{(x-c)^5}{60} + \frac{x^5}{60}\Big|_0^a$$

$$= \boxed{\frac{(x+b-c)^5}{180} - \frac{(a+b)^5}{180} - \frac{(a-c)^5}{180} + \frac{a^5}{180}}$$

#9 $f(x,y,z) = x+y$; $W: \; y \le z \le x \, , \; 0 \le y \le x \, , \; 0 \le x \le 1$

$$\int_0^1 \int_0^x \int_y^x x+y \; dz\,dy\,dz$$

$$\int_y^x x+y \; dz = xz + yz \Big|_y^x = (x^2 + xy) - (xy + y^2)$$

$$\int_0^x x^2 - y^2 \, dy = x^2 y - \frac{y^3}{3} \Big|_0^x = x^3 - \frac{x^3}{3}$$

$$\int_0^1 x^3 - \frac{x^3}{3} \, dx = \frac{x^4}{4} - \frac{x^4}{12} \Big|_0^1 = \frac{1}{4} - \frac{1}{12} = \boxed{\frac{1}{6}}$$

#11 $f(x,y,z) = xyz$ $\quad W: \; 0 \le z \le 1 \, , \; 0 \le y \le \sqrt{1-x^2} \, , \; 0 \le x \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 xyz \; dz\,dy\,dx$$

$$xy \int_0^1 z \, dz = xy \cdot \frac{z^2}{2} \Big|_0^1 = xy \left(\frac{1}{2} \right)$$

$$x \cdot \frac{1}{2} \int_0^{\sqrt{1-x^2}} y\,dy = x \cdot \frac{1}{2} \cdot \frac{y^2}{2} \Big|_0^{\sqrt{1-x^2}} = x \cdot \frac{1}{2} \cdot \left(\frac{1-x^2}{2} \right)$$

$$\frac{1}{4} \int_0^1 x - x^3 = \frac{x^2}{2} - \frac{x^4}{4} \Big|_0^1 = \frac{1}{4} \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{1}{8} - \frac{1}{16} = \boxed{\frac{1}{16}}$$

#13 $f(x,y,z) = e^z$; $W: \; x+y+z \le 1 \, , \; x \ge 0 \, , \; y \ge 0 \, , \; z \ge 0$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \; dz\,dy\,dx$$

$$\int_0^{1-x-y} e^z \, dz = e^z \Big|_0^{1-x-y} = e^{1-x-y} - 1$$

$$\int_0^{1-x} e^{1-x-y} - 1 \, dy = -e^{1-x-y} - y \Big|_0^{1-x} = (-e^{\overset{1}{\cancel{1-x-1+x}}} - 1 + x) - (-e^{1-x})$$

$$\int_0^1 -2 + x + e^{1-x} \, dx = -2x + \frac{x^2}{2} - e^{1-x} \Big|_0^1 = \left(-2 + \frac{1}{2} - 1\right) - (-e) = \boxed{-\frac{5}{2} + e}$$

#17 $f(x,y,z) = x \qquad x \geq 0 \quad y \geq 0 \quad z \geq 0 \quad z = y^2 \quad z = 8 - 2x^2 - y^2$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} \int_0^{y^2} x \, dz\, dy\, dx$$

$$y^2 = 8 - 2x^2 - y^2$$
$$2y^2 = 8 - 2x^2$$
$$2y^2 + 2x^2 = 8$$
$$y^2 + x^2 = 4$$
$$r = 2$$
$$y^{\cancel{2}} = \sqrt{4-x^2}$$

$$\int_0^{y^2} x\, dz = xz \Big|_0^{y^2} = xy^2$$

$$\int_0^{\sqrt{4-x^2}} xy^2\, dy = \frac{xy^3}{3} \Big|_0^{\sqrt{4-x^2}} = \frac{x\sqrt{4-x^2}}{3}$$

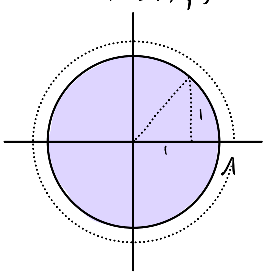
$$\frac{1}{3}\int_0^2 x(4-x^2)^{1/2}\, dx = -\frac{1}{3}\left(-\frac{1}{2}\right)\int_0^4 u^{1/2}\, du = \frac{2}{3}u^{3/2}\Big|_0^4 = \frac{1}{6}\cdot\left(\frac{16}{3}\right) = \frac{16}{18} = \boxed{\frac{8}{9}}$$

$u = 4 - x^2 \quad du = -2x\,dx$

$-\frac{1}{2}du = x\,dx$

15.4 1 5 9 19 27 31 47 51

#1 $f(x,y) = \sqrt{x^2+y^2}, \quad x^2+y^2 \leq 2$

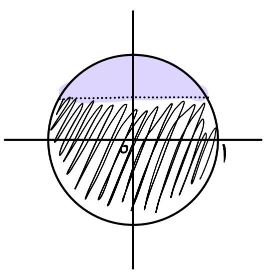


$1^2 + 1^2 = c^2 = \sqrt{2}$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} \sqrt{r^2}\cdot r\, dr\, d\theta \longrightarrow \int_0^{\sqrt{2}} r^2\, dr = \frac{r^3}{3}\Big|_0^{\sqrt{2}} = \frac{(\sqrt{2})^3}{3}$$

$$\frac{2^{3/2}}{3}\int_0^{2\pi} d\theta = \frac{2^{3/2}}{3}\cdot \theta \Big|_0^{2\pi} = \boxed{\frac{2^{5/2}\pi}{3}}$$

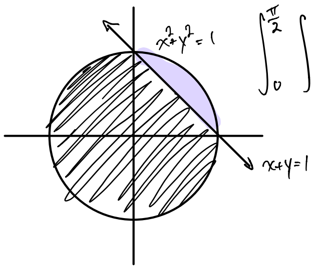
#5 $f(x,y) = y(x^2+y^2)^{-1}; \quad y \geq \frac{1}{2}, \quad x^2+y^2 \leq 1$



$$\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^1 \frac{\cancel{r}\sin\theta}{\cancel{r}}\, r\, dr\, d\theta = \sin\theta \frac{r^2}{2}\Big|_0^1 = \sin\theta \cdot \frac{1}{2}$$

$$\frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sin\theta\, d\theta = \frac{1}{2}\cdot\left(-\cos\theta \Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}\right) = \frac{1}{2}\left(\frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = \boxed{\frac{\sqrt{3}}{2}}$$

#19 $f(x,y) = x - y \quad x^2+y^2 \leq 1 \quad x+y \geq 1$



$1^2 + 1^2$

$$\int_0^{\frac{\pi}{2}}\int_0^1 (r\cos\theta - r\sin\theta)\cdot r\, dr\, d\theta$$

$$\int_0^1 r^2(\cos\theta - \sin\theta)\, dr = \frac{r^3}{3}\Big|_0^1 = (\cos\theta - \sin\theta)\cdot\frac{1}{3}$$

$$\frac{1}{3}\int_0^{\pi/2} \cos\theta - \sin\theta\, d\theta = \sin\theta + \cos\theta\Big|_0^{\pi/2} = \frac{1}{3}\cdot[(1+0)-(0+1)] = \boxed{0}$$

#27 $f(x,y,z) = x^2+y^2$; $x^2+y^2 \le 9$, $0 \le z \le 5$

$r=3$

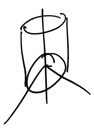
$$\int_0^{2\pi}\int_0^{2\pi}\int_0^5 r^2 \cdot r \, dz\, dr\, d\theta$$

$$\int_0^5 r^3\,dz = r^3 z \Big|_0^5 = 5r^3$$

$$\int_0^{2\pi} 5r^3\,dr = \frac{5r^4}{4}\Big|_0^{2\pi} = \frac{5(2\pi)^4}{4}$$

$$\int_0^{2\pi} \frac{5(2\pi)^4}{4}\,d\theta = \frac{5(2\pi)^5}{20} = \boxed{\frac{(2\pi)^5}{5}}$$

#31 $f(x,y,z) = z$; $x^2+y^2 \le z \le 9$ $\quad r^2 \le z \le 9$



$$\int_0^{2\pi}\int_0^3\int_{r^2}^9 zr \cdot dz\,dr\,d\theta$$

$$\int_{r^2}^9 z \cdot r\,dz = \frac{rz^2}{2}\Big|_{r^2}^9 = \frac{r81}{2} - \frac{r^5}{2}$$

$$\frac{1}{2}\int_0^3 81r - r^5\,dr = \frac{81r^2}{2} - \frac{r^6}{6}\Big|_0^3 = \left(\frac{81\cdot 9}{2} - \frac{3^6}{6}\right) = \frac{729}{2} - \frac{243}{2} = \frac{243}{2}$$

$$\int_0^{2\pi} \frac{243}{2}\,d\theta = \frac{243\theta}{2}\Big|_0^{2\pi} = \frac{486\pi}{2} = \boxed{243\pi}$$

#47 $f(x,y,z) = x^2 + y^2$, $\rho \le 1$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^1 (\rho^2\sin^2\varphi\cos^2\theta + \rho^2\sin^2\varphi\sin^2\theta)\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$\rho^4\sin^3\varphi\cos^2\theta + \rho^4\sin^3\varphi\sin^2\theta \; d\rho\,d\theta\,d\varphi$$

$$\int_0^1 \rho^4\sin^3\varphi\cos^2\theta + \rho^4\sin^3\varphi\sin^2\theta\,d\rho = \frac{1}{5}\cdot\rho^6\sin^3\varphi\cos^2\theta + \rho^5\sin^3\varphi\sin^2\theta\Big|_0^1 = \frac{1}{5}(\sin^3\varphi\cos^2\theta + \sin^3\varphi\sin^2\theta)$$

$$\frac{\sin^3\varphi}{5}\int_0^{2\pi}\cos^2\theta + \sin^2\theta\,d\theta = \frac{\sin^2\theta}{2} - \frac{\cos^2\theta}{2}\Big|_0^{2\pi} = \frac{\sin^3\varphi}{5}\left(\left(0-\frac{1}{2}\right) - \left(0 - \frac{1}{2}\right)\right) = 0$$

$$\int_0^{\pi} 0\,d\varphi = \boxed{0}$$

#51 $$\int_0^{\pi/2}\int_0^{\pi/3}\int_1^2 \rho\cos\varphi\,\rho^2\sin\varphi\,d\rho\,d\theta\,d\varphi$$

$$\int_1^2 \rho^3\cos\phi\sin\phi\,d\rho = \cos\phi\sin\phi\cdot\frac{\rho^4}{4}\Big|_1^2 = \cos\phi\sin\phi\cdot\left(4\cdot\frac{1}{4}\right) = \cos\phi\sin\phi$$

$$\int_0^{\pi/3}\cos\phi\sin\phi\,d\theta = \theta\cos\phi\sin\phi\Big|_0^{\pi/3} = \frac{\pi}{3}\cos\phi\sin\phi$$

$$\frac{\pi}{3}\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi = \frac{\pi}{3}\int_0^1 u\,du = \frac{\pi}{3}\cdot\frac{u^2}{2}\Big|_0^1 = \frac{\pi}{3}\cdot\frac{1}{2} = \boxed{\frac{\pi}{6}}$$

$u = \sin\phi \quad du = \cos\phi\,d\phi$