15.3 Homework - # 3, 5, 7, 9, 11, 13, 15, 17

3. $\int_0^1 \int_0^1 \int_0^2 xe^{y-2z}\, dx\, dy\, dz$

inner int:

$$\int_0^2 xe^{y-2z}\, dx = \left[\frac{1}{2}x^2 e^{y-2z}\right]\Big|_0^2 = \left[\frac{1}{2}(2)^2 e^{y-2z}\right] = 2e^{y-2z}$$

middle int

$$2\int_0^1 e^{y-2z}\, dy = \left| \begin{matrix} u = y-2z \\ du = 1\, dy \end{matrix} \right| = 2\int_0^1 e^u\, du = 2[e^{y-2z}]\Big|_0^1 = 2e^{1-2z} - 2e^{-2z}$$

outer int:

$$\int_0^1 (2(e^{1-2z} - e^{-2z}))\, dz = 2\left[\int_0^1 e^{1-2z} - e^{-2z}\right] dz = \left| \begin{matrix} u1 = 1-2z \\ u2 = -2z \\ du = -2dz \\ \frac{du}{-2} = dz \end{matrix} \right|$$

$$= 2\left[-\frac{1}{2}\int_0^1 e^{u1}\, du + \frac{1}{2}\int_0^1 e^{u2}\, du\right]$$

$$= 2\left[-\frac{1}{2}[e^{1-2z}]\Big|_0^1 + \frac{1}{2}[e^{-2z}]\Big|_0^1\right] = \cancel{2}\left[-\cancel{\frac{1}{2}}(e^{-1} - e^1) + \cancel{\frac{1}{2}}[e^{-2} - 1]\right])$$

$$= [-e^{-1} + e + e^{-2} - 1] = \frac{-1}{e} + e + \frac{1}{e^2} - 1 = (e-1)(1-e^{-2})$$

5. $\int_0^3 \int_0^3 \int_0^1 [(x-y)(y-z)]\, dx\, dy\, dz = \int_0^3 \int_0^3 \int_0^1 (xy - xz - y^2 + yz)\, dx\, dy\, dz$

inner int:

$$\int_0^1 (xy - xz - y^2 + yz)\, dx = \left[\frac{x^2 y}{2} - \frac{x^2 z}{2} - xy^2 + xyz\right]\Big|_0^1 = \left[\frac{1}{2}y - \frac{1}{2}z - y^2 + yz\right]$$

middle int:

$$\int_0^3 \left[\frac{1}{2}y - \frac{1}{2}z - y^2 + yz\right] dy = \left[\frac{y^2}{4} - \frac{1}{2}zy - \frac{y^3}{3} + \frac{y^2 z}{2}\right]\Big|_0^3 = \left[\frac{9}{4} - \frac{3}{2}z - 9 + \frac{9}{2}z\right]$$

outer int

$$\int_0^3 \left[-\frac{3}{2}z + \frac{9}{2}z - \frac{27}{4}\right] dz = \left[-\frac{3}{4}z^2 + \frac{9}{4}z^2 - \frac{27}{4}z\right]\Big|_0^3 = \left[-\frac{27}{4} + \frac{81}{4} - \frac{81}{4}\right] = \frac{-27}{4}$$

7. $\int_0^c \int_0^b \int_0^a (x+z)^3 \, dx\,dy\,dz$

inner int:

$$\int_0^a (x+z)^3 dx = \left| \begin{matrix} u = x+z \\ du = 1dx \end{matrix} \right| = \int_0^a (u)^3 du = \left[\tfrac{1}{4}(u)^4\right]\Big|_0^a = \left[\tfrac{1}{4}(x+z)^4\right]\Big|_0^a$$

$$= \tfrac{1}{4}(a+z)^4 - \tfrac{1}{4}(z)^4$$

middle int:

$$\tfrac{1}{4}\int_0^b ((a+z)^4 - z^4)\, dy = \tfrac{1}{4}\left[y(a+z)^4 - yz^4\right]\Big|_0^b = \tfrac{1}{4}\left[b(a+z)^4 - bz^4\right]$$

outer int:

$$\tfrac{1}{4}b\int_0^c (a+z)^4 - z^4 = \tfrac{1}{4}b\left[\int_0^c (a+z)^4 dz - \int_0^c z^4\right] = \tfrac{1}{4}b\left[\int_0^c u^4 du - \int_0^c z^4 dz\right]$$

$$\left| \begin{matrix} u = (a+z) \\ du = 1\,dz \end{matrix} \right|$$

$$= \tfrac{1}{4}b\left[\tfrac{1}{5}(a+z)^5\Big|_0^c - \tfrac{1}{5}z^5\Big|_0^c\right] = \tfrac{1}{4}b\left[\left(\tfrac{1}{5}(a+c)^5 - \tfrac{1}{5}(a)^5\right) - \left(\tfrac{1}{5}c^5\right)\right]$$

$$= \tfrac{1}{4}b\left[\frac{(a+c)^5}{5} - \frac{a^5}{5} - \frac{c^5}{5}\right] = \frac{b\left[(a+c)^5 - a^5 - c^5\right]}{20} = \frac{b}{20}\left[(a+c)^5 - a^5 - c^5\right]$$

9. $\int_0^1 \int_0^x \int_y^x (x+y)\, dz\,dy\,dx$

inner int: $\int_y^x (x+y)dz = [xz + yz]\Big|_y^x = [x^2 + \cancel{xy}] - [\cancel{xy} + y^2] = x^2 - y^2$

middle int: $\int_0^x (x^2 - y^2)dy = [yx^2 - \tfrac{1}{3}y^3]\Big|_0^x = x^3 - \tfrac{1}{3}x^3$

outer int: $\int_0^1 (x^3 - \tfrac{1}{3}x^3)dx = [x^4 - \tfrac{1}{12}x^4]\Big|_0^1 = [1 - \tfrac{1}{12}] = \tfrac{11}{12}$

1. $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 (xyz)\, dz\, dy\, dx$

inner int:

$\int_0^1 (xyz)dz = xy\left[\frac{z^2}{2}\right]\Big|_0^1 = \frac{1}{2}xy$

middle int:

$\frac{1}{2}x\int_0^{\sqrt{1-x^2}} y\,dy = \frac{1}{2}x\left[\frac{y^2}{2}\right]\Big|_0^{\sqrt{1-x^2}} = \frac{\frac{1}{2}x(1-x^2)}{2}$

$= \frac{1}{4}x - \frac{1}{4}x^3$

outer int:

$\frac{1}{4}\int_0^1 x - x^3\, dx = \frac{1}{4}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]\Big|_0^1 = \frac{1}{4}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{1}{4}\left[\frac{1}{4}\right] = \frac{1}{16}$

13. $x+y+z \leq 1$

$x+y+z-1 \leq 0$

$z \leq 1-x-y$

$\int_0^1 \int_0^1 \int_0^{1-x-y} e^z\, dz\, dy\, dx$

inner int: $[e^z]\Big|_0^{1-x-y} = e^{1-x-y} - 1$

middle int: $\int_0 e^{1-x-y} - 1\, dy = \left|\begin{matrix} u = 1-x-y \\ du = -dy \end{matrix}\right| = -[e^{1-x-y}]\Big|_0^1 - [y]\Big|_0^1$

outer int: $\int_0^1 (-e^{-x} + e^{1-x} - 1)\, dx$

$= -[e^{-x} - e^{1-x}] - 1$

$= -e^{-x} + e^{1-x} - 1$

15. $\int_0^3 \int_0^1 \int_x^{\sqrt{9-x^2-z^2}} (z)\, dy\, dx\, dz$

inner int:

$[zy]\Big|_x^{\sqrt{9-x^2-z^2}} = z\sqrt{9-x^2-z^2} - xz$

middle int:

$\int_0 \left(z\sqrt{9-x^2-z^2} - xz\right) dx = z\left[\int_0^1 \sqrt{9-x^2-z^2} - \left[\frac{x^2}{2}\right]\Big|_0^1\right]$

$\left|\begin{matrix} u = 9-x^2-z^2 \\ du = -2x\,dx \end{matrix}\right.$

17. $y^2 = 8 - 2x^2 - y^2$

$2y^2 = 8 - 2x^2$

$y = \sqrt{4 - x^2}$

$\sqrt{4 - x^2} = 0$

$x = -2, +2$

$$\int_0^2 \int_0^{\sqrt{4-x^2}} (x)\, dy\, dx$$

inner int: $\int_0^{\sqrt{4-x^2}} x\, dy = xy \Big|_0^{\sqrt{4-x^2}} = x\sqrt{4-x^2}$

outer int:

$$\int_0^2 x\sqrt{4-x^2} = -\frac{1}{2}\int u^{1/2}\, du = -\frac{1}{2}\left(\frac{2}{3}u^{3/2}\right)\Big|_0^2$$

$u = 4 - x^2$

$du = -2x\,dx$

$\frac{du}{-2} = x\,dx$

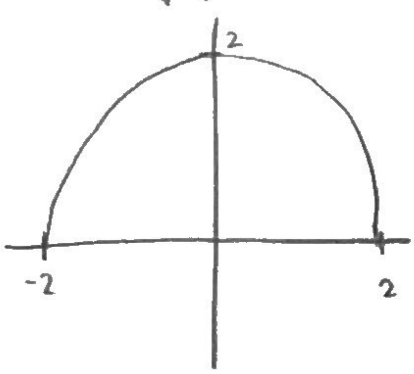
$$= -\frac{1}{3}\left[(4-x^2)^{3/2}\right]\Big|_0^2$$

$$= -\frac{1}{3}\left[-(4)^{3/2} = \frac{8}{3}\right.$$

15.4 HW - #1, 5, 9, 19, 27, 31, 47, 51

1. $f(x,y) = \sqrt{x^2+y^2}$ $\quad x^2+y^2 \leq 2$

$\sqrt{r}$ $\quad r \leq 2$



$$\int_0^{\pi} \int_0^2 (\sqrt{r})\, r\, dr\, d\theta$$

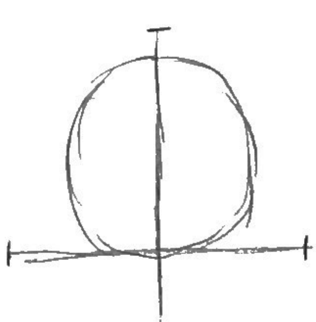
inner integral: $\int_0^2 r^{3/2}\, dr = \frac{2}{5} r^{5/2} \Big|_0^2 = \frac{2}{5}(2)^{5/2}$

outer integral: $\int_0^{\pi} \frac{2}{5}(2)^{5/2}\, d\theta = \frac{2}{5}(2)^{5/2}\theta \Big|_0^{\pi}$

$$= \frac{2}{5}(2)^{5/2}\pi$$

5. $f(x,y) = y(x^2+y^2)^{-1}$, $y \geq \frac{1}{2}$, $x^2+y^2 \leq 1$

$r \leq 1$



$$\int_0^{2\pi} \int_0^1 \frac{(r\sin\theta)}{r}\, r\, dr\, d\theta$$

$f(x,y) = r\sin\theta(r)^{-1}$

inner int: $\int_0^1 \sin\theta\, r\, dr = \sin\theta\left[\frac{r^2}{2}\right]\Big|_0^1 = \frac{1}{2}\sin\theta$

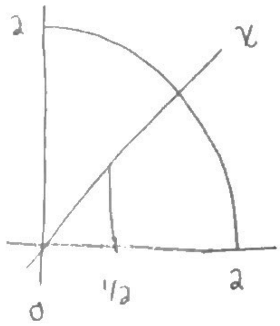
outer int: $\frac{1}{2}\int_0^{2\pi} \sin\theta\, d\theta = \frac{1}{2}\left[\cos\theta\right]\Big|_0^{2\pi}$

$$= \frac{1}{2}\left[-\cos(2\pi) + \cos(0)\right]$$

19 27

9. $\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x \, dy dx$ $\quad \int_0^{\frac{\pi}{2}} \int_0^{2} r^2\cos\theta \, dr d\theta$

inner int: $\cos\theta \int_0^2 r^2 dr = \cos\theta \left[\frac{r^3}{3} \right]\Big|_0^2$



$= \cos\theta \left[\frac{8}{3} \right]$

outer int: $\frac{8}{3}\int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{8}{3}\left[-\sin\theta \right]\Big|_0^{\frac{\pi}{2}}$

$= \frac{8}{3}\left[-\sin(\frac{\pi}{2}) + 0 \right] = -\frac{8}{3}\sin(\frac{\pi}{2})$

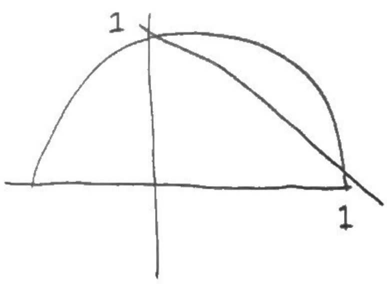
19. $f(x,y) = x - y$ $\quad \int_0^{\frac{\pi}{2}} \int_0^{1} (r\cos\theta - r\sin\theta) r \, dr d\theta$

$x^2 + y^2 \leq 1$

inner int: $\int_0^1 r^2\cos\theta - r^2\sin\theta \, dr$

$x + y \geq 1$

$= \left[\frac{r^3}{3}\cos\theta - \frac{r^3}{3}\sin\theta \right]\Big|_0^1$



$= \frac{1}{3}\left[\cos\theta \cdot \sin\theta \right]$

outer int:

$\frac{1}{3}\int_0^{\frac{\pi}{2}} \cos\theta - \sin\theta \, d\theta = \frac{1}{3}\left[-\sin\theta + \cos\theta \right]\Big|_0^{\frac{\pi}{2}}$

$= \frac{1}{3}\left(\left[-\sin(\frac{\pi}{2}) + \cos(\frac{\pi}{2}) \right] - \left[-\sin(0) + \cos(0) \right] \right)$

$= \frac{1}{3}\left([-1] - [1] \right) = \frac{-2}{3}$

27. $\int_0^{2\pi}\int_0^9 (r^2) dr d\theta$

$\left[\frac{r^3}{3} \right]\Big|_0^9 = 243$

$\int_0^{2\pi} (243) d\theta$

$243(2\pi - 0) = 486\pi$

31. $f(x,y,z) = z$ $\quad \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} F(r,\theta) dr d\theta$

$x^2 + y^2 \leq z \leq 9$

$r \leq z \leq 9$

47. $f(x,y,z) = x^2 + y^2$; $\rho \leq 1$

$= r$

$$\int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta$$

$$\left[\frac{r^3}{3} \right] \Big|_0^1 = \frac{1}{3}$$

$$\int_0^{2\pi} (1) \, d\theta = \frac{1}{3}(2\pi) = \frac{2}{3}\pi$$

51. $0 \leq \theta \leq \frac{\pi}{3}$ $\quad f(x,y,z) = z$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_1^2 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$0 \leq \phi \leq \frac{\pi}{2}$

inner int: $\int_1^2 \rho^3 \cos\phi \sin\phi \, d\rho = \frac{\rho^4}{4} \cos\phi \sin\phi \Big|_1^2$

$1 \leq \rho \leq 2$

$= 4\cos\phi \sin\phi - \frac{1}{4}\cos\phi\sin\phi = \frac{15}{4}\cos\phi\sin\phi$

middle int: $\int_0^{\frac{\pi}{3}} \left(\frac{15}{4}\cos\phi\sin\phi\right) d\theta = \frac{15}{4}\cos\phi\sin\phi\,\frac{\pi}{3} = \frac{15\pi}{12}\cos\phi\sin\phi$

outer int: $\int_0^{\frac{\pi}{2}} \frac{5\pi}{4}\cos\phi\sin\phi \Big|_0^{\frac{\pi}{2}} = \frac{5\pi}{4}\left[\frac{1}{2}\sin^2\phi\right]\Big|_0^{\frac{\pi}{2}} = \frac{5\pi}{4}\left(\frac{1}{2}\sin^2\left(\frac{\pi}{2}\right) - \frac{1}{2}\sin(0)\right)$

$\left| \begin{array}{l} u = \sin\phi \\ du = \cos\phi \, d\phi \end{array} \right|$

$= \frac{5\pi}{8}\sin^2\left(\frac{\pi}{2}\right)$