15.3 Homework

3. $$\int_0^2\int_0^1\int_0^1 xe^{y-2z}\,dz\,dy\,dx$$

$$= \int_0^2\int_0^1 \left(\frac{xe^{y}}{2} - \frac{xe^{y-2}}{2}\right) dy\,dx$$

$$= \int_0^2 \left(\frac{xe}{2} - \frac{xe^{-1}}{2}\right) dx = \frac{1}{2}\int_0^2 xe - xe^{-1}\,dx$$

$$= \frac{1}{2}\left[\frac{ex^2}{2} - \frac{x^2e^{-1}}{2}\right]_0^2$$

$$= \frac{1}{4}\left(4e - 4e^{-1}\right)$$

5. $$\int_0^3\int_0^3\int_0^1 (xy - xz - y^2 + yz)\,dx\,dy\,dz$$

$$= \int_0^3\int_0^3 \left(\frac{y}{2} - \frac{z}{2} - y^2 + yz\right) dy\,dz$$

$$= \int_0^3 \left(\frac{9}{4} - \frac{3z}{2} - 9 + \frac{9z}{2}\right) dz$$

$$= \frac{3z^2}{2} - \frac{27z}{4}\Big|_0^3$$

$$= -\frac{27}{4}$$

7. $(x+z)^3$

$$\int_0^c\int_0^b\int_0^a (x+z)^3\,dx\,dy\,dz$$

$$= \int_0^c\int_0^b \left(\frac{(a+z)^4}{4} - \frac{(0+z)^4}{4}\right) dy\,dz$$

$$= \int_0^c \left(\frac{(a+z)^4 b}{4} - \frac{bz^4}{4}\right) dz$$

$$= \frac{(a+z)^5 b}{20} - \frac{bz^5}{20}\Bigg|_0^c$$

$$= \frac{(a+c)^5 b}{20} - \frac{b(z)^5}{20}$$

9. $x+y$

$y \le t \le x \qquad 0 \le y \le x \qquad 0 \le x \le 1$

$$\int_0^1\int_0^x\int_y^x (x+y)\,dz\,dy\,dx$$

$$= \int_0^1\int_0^x (x^2-y^2)\,dy\,dx$$

$$= \int_0^1 \frac{2}{3}x^3\,dx = \frac{x^4}{6}\Bigg|_0^1 = \frac{1}{6}$$

11. xyz $\quad 0 \le z \le 1 \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1$

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^1 xyz \, dz \, dy \, dx$$

$$= \frac{xy}{2}$$

$$\int_0^{\sqrt{1-x^2}} \frac{xy}{2} \, dy = \frac{xy^2}{4} \Big|_0^{\sqrt{1-x^2}} = \frac{x - x^3}{4}$$

$$\int_0^1 \frac{x - x^3}{4} \, dx = \frac{1}{4} \int_0^1 x - x^3 = \left(\frac{x^2}{2} - \frac{x^4}{4}\right) \frac{1}{4} = \frac{x^2}{8} - \frac{x^4}{32} \Big|_0^1$$

$$= \frac{1}{16}$$

13. e^z $\quad x+y+z \le 1 \quad x \ge 0 \quad y \ge 0 \quad z \ge 0$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} e^z \, dz \, dy \, dx = \int_0^1 \int_0^{1-x} e^{1-x-y} - 1 \, dy \, dx$$

$$= \int_0^1 x - 2 + e^{1-x} \, dx = \frac{x^2}{2} - 2x - e^{1-x} \Big|_0^1$$

$$\left[\frac{1}{2} - 2 - 1\right] + e$$

15. z, radius $= 3$ $\quad x=1,\ y=0,\ x=y$

$x^2+y^2+z^2=9$

$0 \le x \le 1 \qquad 0 \le y \le x \qquad 0 \le z \le \sqrt{9-x^2-y^2}$

$$\int_0^1\int_0^x\int_0^{\sqrt{9-x^2-y^2}} z\,dz\,dy\,dx = \int_0^1\int_0^x \frac{9-x^2-y^2}{2}\,dy\,dx$$

$$= \int_0^1 \frac{9x-x^3-\frac{1}{3}x^3}{2}\,dx = \frac{1}{2}\int_0^1 9x-x^3-\frac{1}{3}x^3$$

$$= \frac{1}{2}\cdot\left[\frac{9x^2}{2}-\frac{x^4}{3}-\frac{x^4}{12}\right]_0^1$$

$$= \frac{1}{2}\left[\frac{27}{16}\right] = \frac{27}{12}$$

17. x, $\quad x \ge 0 \quad y \ge 0 \quad z \ge 0 \quad$ above $z=y^2$

$z = 8-2x^2-y^2$

$y^2 = 8-2x^2-y^2$

$2x^2+2y^2=8$

$x^2+y^2=4$

$0 \le y \le 2, \quad 0 \le x \le \sqrt{4-y^2} \qquad y^2 \le z \le 8-2x^2-y^2$

$$\int_0^2\int_0^{\sqrt{4-y^2}}\int_{y^2}^{8-2x^2-y^2} x\,dz\,dx\,dy = x\cdot z\Big|_{y^2}^{8-2x^2-y^2}$$

$x((8-2x^2-y^2)-(y^2))$

$\cancel{xz}$

$$\int_0^2\int_0^{\sqrt{4-y^2}} x(8-2x^2-2y^2)\,dx\,dy \qquad 8-2x^2-y^2-y^2$$

$$\cancel{\int_0^2 4-\frac{1}{2}(-y^2+4)^2+\frac{1}{2}(-2y^2+8)x^2}$$

$$\int_0^2 \left(-\frac{1}{2}x^4 + \frac{1}{2}(-2y^2+8)\, x(-y^2+4) \right) dy$$

$$= \frac{1}{10}y^2 - \frac{4}{3}y^3 + 8y \Big|_0^2 = \frac{128}{5}$$

15.4 Homework

1. $f(x,y) = \sqrt{x^2+y^2} \quad x^2+y^2 \le 2$

$$R = \sqrt{2}$$

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta$

$$0 \le \theta \le 2\pi \quad 0 \le r \le \sqrt{2}$$

$$\int_0^{2\pi}\int_0^{\sqrt{2}} r(r\,dr\,d\theta) = \frac{r^3}{3}\Big|_0^{\sqrt{2}} = \frac{1}{3}\int_0^{2\pi} (\sqrt{2})^3\,d\theta$$

$$\frac{2\sqrt{2}}{3} \cdot \theta\Big|_0^{2\pi}$$

$$= \frac{4\sqrt{2}\pi}{3}$$

5. $f(x,y) = y(x^2+y^2)^{-1} \quad y \ge 1/2 \quad x^2+y^2 \le 1$

$x = r\cos\theta \quad y = r\sin\theta \quad dx\,dy = r\,dr\,d\theta \quad x^2+y^2=1,\ r=1$

$$y = \frac{1}{2} \quad r = \frac{1}{2\sin\theta} \quad 1 = \frac{1}{2\sin\theta}$$

$$\sin\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$\frac{1}{2\sin\theta} \le r \le 1 \qquad \frac{\pi}{6} \le \theta \le \frac{5\pi}{6}$$

$$\int_{\pi/6}^{5\pi/6}\int_{\frac{1}{2\sin\theta}}^{1} \frac{r\sin\theta}{r^2}\, r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\int_{\frac{1}{2\sin\theta}}^{1} \sin\theta\,dr\,d\theta$$

$$= \sin\theta\, r\Big|_{\frac{1}{2\sin\theta}}^{1} = \frac{1}{2}\int_{\pi/6}^{5\pi/6} (2\sin\theta - 1)\,d\theta = \frac{-2\cos\theta - \theta}{2}\Big|_{\frac{\pi}{6}}^{\frac{5\pi}{6}}$$

9. $\int_0^{1/2} \int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\,dy\,dx$ $\quad 0 \le x \le \frac{1}{2} \quad \sqrt{3}x \le y \le \sqrt{1-x^2}$

$\sqrt{3}x = \sqrt{1-x^2}$

$x = \frac{1}{2}$

$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

$\tan^{-1}\left(\sqrt{3}/2 \,/\, 1/2\right) \quad \theta = \frac{\pi}{3}$

$dy\,dx = r\,dr\,d\theta$

$y \le \sqrt{1-x^2}$

$y^2 \le 1 - x^2$

$x^2 + y^2 \le 1$

$\int_{\pi/3}^{\pi/2} \int_0^1 (r\cos\theta)\, r\,dr\,d\theta \quad r \le 1 \quad \theta = \pi/2$

$\int_0^1 r^2 \cos\theta\, dr = \frac{r^3}{3}\cos\theta \Big|_0^1 \quad \int_{\pi/3}^{\pi/2} \frac{\cos\theta}{3}\, d\theta$

$\frac{1}{3}\int \cos\theta\, d\theta = \frac{\sin\theta}{3}\Big|_{\pi/3}^{\pi/2} = \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right)$

19. $f(x,y) = x - y$

$x^2 + y^2 \le 1 \quad y + x \ge 1$

$x^2 + y^2 = x + y \rightarrow (1,0) \quad (0,1)$

$x^2 + y^2 = 1 \quad r = 1$

$x + y = 1$

$r = \frac{1}{\cos\theta + \sin\theta}$

$0 \le \theta \le \frac{\pi}{2}$ $\quad$ ~~r ≤ r ≤~~

$\frac{1}{\cos\theta + \sin\theta} \le r \le 1$

$$\int_0^{\frac{\pi}{2}} \int_{\frac{1}{\cos\theta+\sin\theta}}^{1} (r\cos\theta - r\sin\theta)\, r\, dr\, d\theta$$

$$r^2(\cos\theta - \sin\theta)\, dr$$

$$\frac{r^3}{3}(\cos\theta - \sin\theta) \Big|_{\frac{1}{\cos\theta+\sin\theta}}^{1}$$

$$\frac{1}{3}\int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{(\cos\theta+\sin\theta)^3}\right) (\cos\theta - \sin\theta\, d\theta$$

Using maple I get 0

27. $f(x,y,z) = x^2+y^2$; $x^2+y^2 \le 9$, $0 \le z \le 5$

$$x^2+y^2 \le 9$$

$$(r\cos\theta)^2 + (r\sin\theta)^2 \le 9$$

$$r^2(\cos^2\theta + \sin^2\theta) \le 9$$

$$r^2 \le 9$$

$$0 \le r \le 3$$

$$0 \le \theta \le 2\pi$$

$$0 \le z \le 5$$

$$\int_0^{2\pi}\int_0^{3}\int_0^{5} (r^2)\, r\, dz\, dr\, d\theta = r^3\, dz = zr^3 \Big|_0^5$$

$$\int_0^3 5r^3\, dr = \frac{5r^4}{4}\Big|_0^3 = \frac{81}{4}$$

$$\int_0^{2\pi} \frac{81}{4}\, d\theta = \frac{81\theta}{4}\Big|_0^{2\pi} = \boxed{\frac{405\pi}{2}}$$

31. $f(x,y,z) = z;\quad x^2+y^2 \le z \le 9$

$$x^2+y^2 = 9$$
$$r^2 = 9$$
$$r = 3$$

$0 \le \theta \le 2\pi \qquad 0 \le r \le 3 \qquad r^2 \le z \le 9$

$$\int_0^{2\pi}\int_0^{3}\int_{r^2}^{9} z\, r\, dz\, dr\, d\theta = \quad r \cdot \frac{z^2}{2}\Big|_{r^2}^{9}$$

$$\int_0^3 81r - r^5\, dr$$

$$= \frac{81r^2}{2} - \frac{r^6}{6}\Big|_0^3$$

$$= \frac{1}{2}\int_0^{2\pi} 243\, d\theta = \frac{243\theta}{2}\Big|_0^{2\pi}$$

$$= \boxed{243\pi}$$

47. $f(x,y,z) = x^2+y^2;\quad \rho \le 1$

$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi$

$dV = \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$

$0 \le \theta \le 2\pi \qquad 0 \le \phi \le \pi \qquad 0 \le \rho \le 1$

$$x^2+y^2$$
$$= \rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi$$
$$= \rho^2\sin^2\phi$$

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{1}\left(\rho^2\sin^2\phi\right)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \int_0^1 \rho^4\, d\rho \cdot \int_0^{\pi} \sin^3\phi\, d\phi \cdot \int_0^{2\pi} d\theta$$

$$\frac{\rho^5}{5}\Big|_0^1 \cdot \frac{\cos^3\phi - \cos\phi}{3}\Big|_0^{\pi} \cdot \theta\Big|_0^{2\pi}$$

$$= \frac{8\pi}{15}$$

51. $f(x,y,z) = z$ $\quad 0 \le \theta \le \pi/3$

$0 \le \phi \le \pi/2$

$1 \le \rho \le 2$

$z = \rho\cos\phi \qquad dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

$$\int_0^{\pi/3}\int_0^{\pi/2}\int_1^{2} (\rho\cos\phi)\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_0^{\pi/3}\int_0^{\pi/2}\int_1^{2} \rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta$$

$$\int_1^2 \rho^3 \, d\rho = \frac{\rho^4}{4}\Big|_1^2 = \frac{16-1}{4} = \frac{15}{4}$$

$$\int_0^{\pi/2} \sin\phi\cos\phi \, d\phi = \frac{\sin^2\phi}{2}\Big|_0^{\pi/2} = \frac{1}{2}$$

$$\int_0^{\pi/3} d\theta = \theta\Big|_0^{\pi/3} = \frac{\pi}{3}$$

$$\frac{15}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{3} = \boxed{\frac{15\pi}{24}}$$