15.3, 15.4 HW

10/26/20

<u>15.3</u> : # 3, 5, 7, 9, 11, 13, 15, 17

3. $f(x,y,z) = xe^{y-2z}$; $0 \le x \le 2$, $0 \le y \le 1$, $0 \le z \le 1$

$$\iiint_B xe^{y-2z}\, dV = \int_0^2\int_0^1\int_0^1 xe^{y-2z}\, dz\,dy\,dx = \left(\int_0^2 x\,dx\right)\left(\int_0^1 e^y\,dy\right)\left(\int_0^1 e^{-2z}\,dz\right)$$

$$= \left(\frac{1}{2}x^2\Big|_0^2\right)\left(e^y\Big|_0^1\right)\left(-\frac{1}{2}e^{-2z}\Big|_0^1\right) = 2(e-1)\cdot -\frac{1}{2}(e^{-2}-1) = (e-1)(1-e^2)$$

5. $f(x,y,z) = (x-y)(y-z)$; $[0,1]\times[0,3]\times[0,3]$

$$\iiint_B (x-y)(y-z)\,dV = \int_0^1\int_0^3\int_0^3 (x-y)(y-z)\,dz\,dy\,dx = \int_0^1\int_0^3\left(\int_0^3 (x-y)(y-z)\,dz\right)dy\,dx$$

$$= \int_0^1\int_0^3 (x-y)\left(yz-\frac{1}{2}z^2\right)\Big|_{z=0}^3 dy\,dx = \int_0^1\int_0^3 (x-y)\left(3y-\frac{9}{2}\right)dy\,dx$$

$$= \int_0^1\int_0^3\left(\left(3x+\frac{9}{2}\right)y-\frac{9}{2}x-3y^2\right)dy\,dx = \int_0^1\left(\frac{3}{2}x+\frac{9}{4}\right)y^2-\frac{9}{2}xy-y^3\Big|_{y=0}^3 dx$$

$$= \int_0^1\left(\left(\frac{3}{2}x+\frac{9}{4}\right)\cdot 9-\frac{9}{2}x\cdot 3-27\right)dx = \int_0^1 -\frac{27}{4}\,dx = -\frac{27}{4} = -6.75$$

7. $f(x,y,z) = (x+z)^3$; $[0,a]\times[0,b]\times[0,c]$

$$\iiint_B f(x,y,z)\,dV = \int_0^a\int_0^b\int_0^c (x+z)^3\,dz\,dy\,dx = \int_0^a\int_0^b \frac{(x+z)^4}{4}\Big|_{z=0}^c dy\,dx$$

$$= \int_0^a\int_0^b\left(\frac{(x+c)^4}{4}-\frac{x^4}{4}\right)dy\,dx = \int_0^a \frac{(x+c)^4-x^4}{4}y\Big|_{y=0}^b dx$$

$$= \int_0^a \frac{b}{4}\left[(x+c)^4-x^4\right]dx = \frac{b}{4}\left[\frac{(x+c)^5}{5}-\frac{x^5}{5}\right]\Big|_{x=0}^a = \frac{b}{4}\,\frac{(a+c)^5-a^5-c^5}{5}$$

$$= \frac{b}{20}\left[(a+c)^5-a^5-c^5\right]$$

9. $f(x,y,z) = x+y$; $W: y \le z \le x$, $0 \le y \le x$, $0 \le x \le 1$

$$\iiint_W (x+y)\,dV = \iint_D\left(\int_y^x (x+y)\,dz\right)dA = \iint_D (x+y)z\Big|_{z=y}^x dA = \iint_D (x+y)(x-y)\,dA$$

$$= \iint_D (x^2-y^2)\,dA = \int_0^1\int_0^x (x^2-y^2)\,dy\,dx = \int_0^1\left(\int_0^x (x^2-y^2)\,dy\right)dx$$

$$= \int_0^1 x^2y-\frac{y^3}{3}\Big|_{y=0}^x dx = \int_0^1 \frac{2x^3}{3}\,dx = \frac{2}{12}x^4\Big|_0^1 = \frac{1}{6}$$

11. $f(x,y,z) = xyz$; $W: 0 \le z \le 1, \quad 0 \le y \le \sqrt{1-x^2}, \quad 0 \le x \le 1$

$$\iiint_W xyz\,dV = \iint_D \left(\int_0^1 xyz\,dz\right) dA = \iint_D \frac{xyz^2}{2}\Big|_{z=0}^{1} dA = \iint_D \frac{xy}{2}\,dA$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \frac{xy}{2}\,dy\right) dx = \int_0^1 \frac{xy^2}{4}\Big|_{y=0}^{\sqrt{1-x^2}} dx = \int_0^1 \frac{x(1-x^2)}{4}\,dx = \int_0^1 \frac{x-x^3}{4}\,dx$$

$$= \frac{x^2}{8} - \frac{x^4}{16}\Big|_0^1 = \frac{1}{8} - \frac{1}{16} = \frac{1}{16}$$

13. $f(x,y,z) = e^z$; $W: x+y+z \le 1, \quad x \ge 0, \quad y \ge 0, \quad z \ge 0$

$$\begin{matrix} x+y+z = 1 \\ z = 0 \end{matrix} \quad \Rightarrow \quad x+y = 1$$

$$\iiint_W e^z\,dV = \iint_D \left(\int_0^{1-x-y} e^z\,dz\right) dA = \iint_D e^z\Big|_{z=0}^{1-x-y} dA = \iint_D (e^{1-x-y} - 1)\,dA$$

$$= \int_0^1 \left(\int_0^{1-y} (e^{1-x-y} - 1)\,dx\right) dy = \int_0^1 (-e^{1-x-y} - x)\Big|_{x=0}^{1-y} dy$$

$$= \int_0^1 \left(-e^{1-(1-y)-y} - (1-y) + e^{1-0-y} + 0\right) dy = \int_0^1 (y - 2 + e^{1-y})\,dy$$

$$= \left(\frac{1}{2}y^2 - 2y - e^{1-y}\right)\Big|_0^1 = e - \frac{5}{2}$$

15. $$\iiint_V z\,dV = \iint_D \left(\int_0^{\sqrt{9-x^2-y^2}} z\,dz\right) dA = \iint_D \frac{z^2}{2}\Big|_0^{\sqrt{9-x^2-y^2}} dA = \iint_D \frac{9-x^2-y^2}{2}\,dA$$

$$= \int_0^1 \left(\int_0^x \frac{9-x^2-y^2}{2}\,dy\right) dx = \int_0^1 \frac{9y - x^2y - y^3/3}{2}\Big|_{y=0}^{x} dx = \int_0^1 \left(\frac{9x}{2} - \frac{2x^3}{3}\right) dx$$

$$= \frac{9x^2}{4} - \frac{x^4}{6}\Big|_0^1 = 2\frac{1}{12}$$

17. $$\iiint_W x\,dV = \iint_D \left(\int_{y^2}^{8-2x^2-y^2} x\,dz\right) dA = \iint_D xz\Big|_{y^2}^{8-2x^2-y^2} dA = \iint_D x(8-2x^2-y^2-y^2)\,d$$

$$= \iint_D (8x - 2x^3 - 2xy^2)\,dA = \int_0^2 \int_0^{\sqrt{4-x^2}} (8x - 2x^3 - 2xy^2)\,dy\,dx = \int_0^2 \left(8xy - 2x^3y - \frac{2}{3}xy^3\right)\Big|_0$$

$$= \int_0^2 \left(\frac{16}{3}x\sqrt{4-x^2} - \frac{4}{3}x^3\sqrt{4-x^2}\right) dx \qquad \begin{matrix} u = 4-x^2 \quad \Rightarrow \quad x^2 = 4-u \\ du = -2x\,dx \end{matrix}$$

$$= -\int_4^0 \frac{2}{3}u^{3/2}\,du = -\left(\frac{4}{15}u^{5/2}\right)\Big|_4^0 du = \frac{128}{15}$$

15.4: # 1, 5, 9, 19, 27, 31, 47, 51

1. $f(x,y) = \sqrt{x^2+y^2}$, $x^2+y^2 \le 2$

$D:\quad 0 \le \theta \le 2\pi$, $0 \le r \le \sqrt{2}$

$f(x,y) = \sqrt{x^2+y^2} = \sqrt{r^2} = r$

$$\iint_D \sqrt{x^2+y^2}\, dA = \int_0^{2\pi}\int_0^{\sqrt{2}} r \cdot r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r^2\, dr\, d\theta = \int_0^{2\pi} \frac{r^3}{3}\bigg|_{r=0}^{\sqrt{2}} d\theta$$

$$= \int_0^{2\pi} \frac{(\sqrt{2})^3}{3}\, d\theta = \frac{2\sqrt{2}}{3}\theta\bigg|_0^{2\pi} = \frac{4\sqrt{2}\,\pi}{3}$$

5. $f(x,y) = y(x^2+y^2)^{-1}$; $y \ge 1/2$, $x^2+y^2 \le 1$

$$\alpha = \tan^{-1}\frac{1/2}{\sqrt{3}/2} = \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}$$

$D:\quad \pi/6 \le \theta \le 5\pi/6$, $1/2 \csc\theta \le r \le 1$

$f(x,y) = y(x^2+y^2)^{-1} = (r\sin\theta)(r^2)^{-1} = r^{-1}\sin\theta$

$$\iint_D y(x^2+y^2)^{-1}\, dA = \int_{\pi/6}^{5\pi/6}\int_{1/2\csc\theta}^{1} r^{-1}\sin\theta\, dr\, d\theta = \int_{\pi/6}^{5\pi/6}\int_{1/2\csc\theta}^{1} \sin\theta\, dr\, d\theta$$

$$= \int_{\pi/6}^{5\pi/6} r\sin\theta\bigg|_{r=1/2\csc\theta}^{1} d\theta = \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\sin\theta\csc\theta\right) d\theta = \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\right) d\theta$$

$$= -\cos\theta - \frac{\theta}{2}\bigg|_{\pi/6}^{5\pi/6} = -\cos\frac{5\pi}{6} - \frac{5\pi}{12} - \left(-\cos\frac{\pi}{6} - \frac{\pi}{12}\right) = \frac{\sqrt{3}}{2} - \frac{\pi}{3} + \frac{\sqrt{3}}{2} \approx 0.685$$

9. $\displaystyle\int_0^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\, dy\, dx$; $0 \le x \le \frac{1}{2}$, $\sqrt{3}x \le y \le \sqrt{1-x^2}$

$r\sin\theta = \sqrt{3}\, r\cos\theta \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \pi/3$

$D:\quad \pi/3 \le \theta \le \pi/2$, $0 \le r \le 1$

$$\int_0^{1/2}\int_{\sqrt{3}x}^{\sqrt{1-x^2}} x\, dy\, dx = \int_{\pi/3}^{\pi/2}\int_0^1 r(\cos\theta)\, r\, dr\, d\theta = \int_{\pi/3}^{\pi/2}\int_0^1 r^2\cos\theta\, dr\, d\theta = \int_{\pi/3}^{\pi/2} \frac{r^3\cos\theta}{3}\bigg|_{r=0}^{1} d\theta$$

$$= \int_{\pi/3}^{\pi/2} \frac{\cos\theta}{3}\, d\theta = \frac{\sin\theta}{3}\bigg|_{\pi/3}^{\pi/2} = \frac{1}{3}\left(\sin\frac{\pi}{2} - \sin\frac{\pi}{3}\right) = \frac{1}{3}\left(1 - \frac{\sqrt{3}}{2}\right) \approx 0.045$$

19. $f(x,y) = x - y$; $x^2+y^2 \le 1$, $x + y \ge 1$

$D:\quad 0 \le \theta \le \pi/2$, $1/(\cos\theta + \sin\theta) \le r \le 1$

$f(x,y) = x - y = r\cos\theta - r\sin\theta = r(\cos\theta - \sin\theta)$

$$\iint_D (x-y)\, dA = \int_0^{\pi/2}\int_{1/\cos\theta+\sin\theta}^{1} r(\cos\theta - \sin\theta)\, r\, dr\, d\theta = \int_0^{\pi/2}\int_{1/\cos\theta+\sin\theta}^{1} r^2(\cos\theta - \sin\theta)\, dr\, d\theta$$

$$= \frac{1}{3}\int_0^{\pi/2} r^3(\cos\theta - \sin\theta)\bigg|_{r=\frac{1}{\cos\theta+\sin\theta}}^{1} d\theta = \frac{1}{3}\int_0^{\pi/2} (\cos\theta - \sin\theta)\left(1 - \frac{1}{(\cos\theta+\sin\theta)^3}\right) d\theta$$

$$\frac{1}{3}\left(\int_0^{\pi/2}(\cos\theta-\sin\theta)\,d\theta-\int_0^{\pi/2}\frac{\cos\theta-\sin\theta}{(\cos\theta+\sin\theta)^3}\,d\theta\right)$$

$$\int_0^{\pi/2}(\cos\theta-\sin\theta)\,d\theta=(\sin\theta+\cos\theta)\Big|_0^{\pi/2}=(1+0)-(0+1)=0$$

$$\int_0^{\pi/2}\frac{\cos\theta-\sin\theta}{(\cos\theta+\sin\theta)^3}\,d\theta=\int_1^1\frac{1}{u^3}\,du=0$$

$$\iint_D(x-y)\,dA=0$$

27. $f(x,y,z)=x^2+y^2$; $x^2+y^2\le 9$, $0\le z\le 5$

$D:\ 0\le\theta\le 2\pi$, $0\le r\le 3$

$W:\ 0\le\theta\le 2\pi$, $0\le r\le 3$, $0\le z\le 5$

$$\iiint_W(x^2+y^2)\,dV=\int_0^{2\pi}\int_0^3\int_0^5 r^2\cdot r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^3\int_0^5 r^3\,dz\,dr\,d\theta$$

$$=\left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^3 r^3\,dr\right)\left(\int_0^5 1\,dz\right)=2\pi\cdot 5\cdot\frac{r^4}{4}\Big|_0^3=\frac{5\cdot 3^4\pi}{2}\approx 636.17$$

31. $f(x,y,z)=z$; $x^2+y^2\le z\le 9$

$D:\ 0\le\theta\le 2\pi$, $0\le r\le 3$

$W:\ 0\le\theta\le 2\pi$, $0\le r\le 3$, $r^2\le z\le 9$

$$\iiint_W z\,dV=\int_0^{2\pi}\int_0^3\int_{r^2}^9 zr\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^3\frac{z^2r}{2}\Big|_{z=r^2}^9\,dr\,d\theta=\int_0^{2\pi}\int_0^3\frac{r(81-r^4)}{2}\,dr\,d\theta$$

$$=\int_0^{2\pi}\int_0^3\frac{81r-r^5}{2}\,dr\,d\theta=\int_0^{2\pi}\frac{81r^2}{4}-\frac{r^6}{12}\Big|_0^3\,d\theta=\int_0^{2\pi}121.5\,d\theta=243\pi$$

47. $f(x,y,z)=x^2+y^2$; $\rho\le 1$

$W:\ 0\le\theta\le 2\pi$, $0\le\phi\le\pi$, $0\le\rho\le 1$

$$f(x,y,z)=x^2+y^2=(\rho\cos\theta\sin\phi)^2+(\rho\sin\theta\sin\phi)^2=\rho^2\sin^2\phi(\cos^2\theta+\sin^2\theta)=\rho^2\sin^2\phi$$

$$\iiint_W(x^2+y^2)\,dV=\int_0^{2\pi}\int_0^{\pi}\int_0^1(\rho^2\sin^2\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$=\int_0^{2\pi}\int_0^{\pi}\int_0^1\rho^4\sin^3\phi\,d\rho\,d\phi\,d\theta=\left(\int_0^{2\pi}1\,d\theta\right)\left(\int_0^{\pi}\sin^3\phi\,d\phi\right)\left(\int_0^1\rho^4\,d\rho\right)$$

$$=\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{\sin^2\theta\cos\theta}{3}-\frac{2}{3}\cos\theta\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^1\right)=2\pi\cdot\left(\frac{2}{3}+\frac{2}{3}\right)\cdot\frac{1}{5}=\frac{8\pi}{15}$$

51. $f(x,y,z) = z \;; \quad 0 \le \theta \le \pi/3 \;, \quad 0 \le \phi \le \pi/2 \;, \quad 1 \le \rho \le 2$

$f(x,y,z) = z = \rho\cos\phi$

$$\iiint_W z\,dV = \int_0^{\pi/3}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{\pi/3}\int_0^{\pi/2}\int_1^2 \rho^3 \cos\phi \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \left(\int_0^{\pi/3} 1\, d\theta\right)\left(\int_0^{\pi/2} \frac{1}{2}\sin 2\phi\, d\phi\right)\left(\int_1^2 \rho^3\, d\rho\right) = \frac{\pi}{3}\cdot\left(-\frac{1}{4}\cos 2\phi\right)\Bigg|_0^{\pi/2}\cdot\left(\frac{\rho^4}{4}\Bigg|_1^2\right)$$

$$= \frac{\pi}{3}\cdot\frac{1}{2}\cdot\left(4-\frac{1}{4}\right) = \frac{5}{8}\pi$$