17.1 & 17.2 (Dec. 6th)

17.1: #1, 3, 5, 7, 9, 13

17.2: #1, 3, 5, 9, 11, 13

17.1: #1, 3, 5, 7, 9, 13

1) $x = \cos\theta$, $y = \cos\theta$ $\quad 0 \le \theta \le 2\pi$

$dx = -\sin\theta \, d\theta$, $dy = \cos\theta \, d\theta$

$xy\,dx + y\,dy = (\sin\theta\cos\theta)(-\sin\theta\, d\theta) + \sin\theta\cos\theta\, d\theta$

$= (-\cos\theta\sin^2\theta + \sin\theta\cos\theta)\, d\theta$

$\int_C xy\,dx + y\,dy = \int_0^{2\pi} (-\cos\theta\sin^2\theta + \sin\theta\cos\theta)\, d\theta$

$= -\frac{\sin^3\theta}{3}\Big|_0^{2\pi} - \frac{\cos(2\theta)}{4}\Big|_0^{2\pi} = 0$

$P = xy$, $Q = y$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 - x = -x$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \iint_D -x\,dx\,dy = -\iint_D x\,dx\,dy$

0 over disk due to symmetry.

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \boxed{0}$

3) $P = y^2$, $Q = x^2$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 2y$

$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = \iint_D (2x - 2y)\,dx\,dy$

$= 2\iint_D x\,dx\,dy - 2\iint_D y\,dx\,dy$

by symmetry both are zero.

$\int_C y^2\,dx + x^2\,dy = 0 - 0 = \boxed{0}$

5) $P(x,y) = x^2 y$, $\frac{\partial P}{\partial y} = x^2$

$\frac{\partial Q}{\partial x} = 0$, $Q(x,y) = 0$

$\oint_C x^2 y\,dx = \oint_C (x^2 y\,dx + 0\,dy)$

$\iint (0 - x^2)\,dx\,dy$

$\int_0^{2\pi}\int_0^1 -(r\cos\theta)^2\, r\,dr\,d\theta$

$= -\int_0^{2\pi} \cos^2\theta \left(\int_0^1 r^3 r\,dr \right) d\theta$

$= -\int_0^{2\pi} \cos^2\theta \left(\frac{1}{4} r^4 \right)_0^1 d\theta$

$= -\frac{1}{4} \int_0^{2\pi} \pi \cos^2\theta \, d\theta$

$= -\frac{1}{4} \int_0^{2\pi} \frac{1+\cos(2\theta)}{\cancel{2}} d\theta$

$= -\frac{1}{4} \left[\int_0^{2\pi} \frac{1}{2} d\theta + \int_0^{2\pi} \frac{\cos 2\theta}{2} d\theta \right]$

$= -\frac{1}{4} \left[\frac{2\pi}{2} + 0 \right]$

$= \boxed{-\frac{1}{4}\pi}$

7) $P = Q = x^2$

so, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x - 0 = 2x$

$I = \iint_D 2x\,dA = \int_0^1 \int_{x^2}^{x} 2x\,dy\,dx$

$= \int_0^1 2xy \Big|_{x^2}^{x} dx$

$= \int_0^1 (2x^2 - 2x^3) dx$

$= \frac{2x^3}{3} - \frac{x^4}{2} \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \boxed{\frac{1}{6}}$

9) $P = e^{x+y}, \; Q = e^{x-y}$

$\frac{\partial Q}{\partial x} = e^{x-y}, \; \frac{\partial P}{\partial y} = e^{x+y}$

$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = e^x(e^{-y} - e^y)$

$\int_C F\,ds = \iint_D e^x(e^{-y} - e^y)\,dx\,dy$

$= \int_0^2 \int_y^{y+2} e^x(e^{-y} - e^y)\,dx\,dy = \int_0^2 e^x (e^{-y} - e^y) \Big|_{x=y}^{x=y+2} dy$

$= \int_0^2 (e^{y+2} - e^y)(e^{-y} - e^y)\,dy$

$= \int_0^2 (e^2 - 1)(1 - e^{2y})\,dy$

$= (e^2 - 1)\left(y - \frac{e^{2y}}{2}\right)\Big|_0^2$

$= (e^2 - 1)\left(2 - \frac{e^4}{2} - 0 + \frac{1}{2}\right)$

$= \boxed{(e^2 - 1)\frac{(e^4 - 5)}{2}}$

13) $l = \sin x + y$, $m = (3x+y)$

$\int_c l\,dx + m\,dy = \iint \left(\frac{dm}{dx} - \frac{dl}{dy}\right) dy\,dx$

$= \iint (3-1)\,dy\,dx$

$= 2 \iint dy\,dx$

$= 2 \times$ area of parallelogram

$= 2 \times \left[\frac{1}{2}(9+8) \times 2\right)$

$= \boxed{34}$

17.2: # 1, 3, 5, 9, 11, 13

1) $f = \langle 2xy, x, y+z \rangle$ and surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \le 1$

$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} = i(1-0) - j(0-0) + k(1-2x)$

$= i + (1-2x)k = \langle 1, 0, 1-2x \rangle$

$r_x = \langle 1, 0, -2x \rangle$, $r_y \langle 0, 1, -2y \rangle$

$r_x \times r_y = \begin{vmatrix} i & j & k \\ 1 & 0 & -2x \\ 0 & 1 & -2y \end{vmatrix} = i(-2x) - j(-2y - 0) + k(1-0)$

$= \langle -2x, 2y, 1 \rangle$

$= \iint \langle 1, 0, 1-2x \rangle \cdot \langle -2x, 2y, 1 \rangle \, dA$

$= \iint (-2x + 1 - 2x)\, dA$

$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} (1-4x)\, dx\, dy$

$= \int_0^1 \int_0^{2\pi} (1 - 4r\cos\theta) r\, d\theta\, dr$

$= \int_0^1 \left[r\theta - 4r^2(+\sin\theta)\right]_0^{2\pi} dr$

$= 2\pi \int_0^1 r\,dr = 2\pi \left[\frac{r^2}{2}\right]_0^1 = \boxed{\pi}$

3) along AB, $x=1$, $dx=0$

$\int_{AB} e^{y-2} dx = 0$

Along BC, $y=1$, $z=1$

$\int_{BC} e^{1-1} dx = \int_{BC} dx = \int_1^0 dx = (0-1) = -1$

Along CD, $x=0$, $dx=0$

$\int_{CD} e^{y-2} dx = 0$

along DA, x varies from 0 to 1, $y=0$, $z=1$

$= \int_{DA} e^{0-1} dx = e^{-1} \int_0^1 dx = e^{-1}$

$= -\int_C F \cdot dx = -1 + \frac{1}{e}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y-2} & 0 & 0 \end{vmatrix}$$

$= i(0-0) - j\left(0 - \frac{\partial}{\partial z} e^{y-2}\right) + k\left(0 - \frac{\partial}{\partial y} e^{y-2}\right)$

$\nabla \times F = -e^{y-2} j - e^{y-2} \hat{k}$, have $\vec{n} = k$, $ds = dx\,dy$

$= \int_0^1 \int_0^1 (-e^{y-1})\,dy\,dx$

$= -e^{-1} \int_0^1 \left(\int_0^1 e^{y} dy\right) dx$

$= -1 + \frac{1}{e}$

$\oint_C F\,dx = \iint_S (\nabla \times F) \cdot \hat{n}\,ds$ $\therefore$ Stoke's theorem is verified.

5) $$\mathrm{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{z^2} - y & e^{z^3} + x & (0)(xz) \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y} \cos(xz) - \frac{\partial}{\partial z}(e^{z^3} + x)\right] - j\left[\frac{\partial}{\partial x}(\cos(xz)) - \frac{\partial}{\partial z}(e^{z^2} - y)\right]$

$+ k\left[\frac{\partial}{\partial x}(e^{z^3} + x) - \frac{\partial}{\partial y}(e^{z^2} - y)\right]$

$= [0 - 3z^2e^{z^3}]i - j[-z\sin xz - 2ze^{z^2}] + k(1+1)$

$= -3z^2e^{z^3}\hat{i} + (z\sin xz + 2ze^{z^2})\hat{j} + 2\hat{k}$

$\text{curl} F = -3z^2e^{z^3}\hat{i} + (z\sin(xz) + 2ze^{z^2})\hat{j} + 2\hat{k}$

$C: r(t) = (\cos t, \sin t, 0)$

$\oint_C F \cdot r'(s)\,ds = \int_0^{2\pi} \langle e^{z^2} - y, e^{z^3} + x, \cos(xz)\rangle \cdot \langle -\sin t, \cos t, 0\rangle\,dt$

$= \int_0^{2\pi} \sin^2 t + \cos^2 t + 0\,dt = \int_0^{2\pi} 1\,dt = \boxed{2\pi}$

9) $F = \langle y^2, x^2, xy\rangle$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$= i(x - x) + j(-y + y) + k(z - z)$

$= 0$

$\nabla \times F = 0$

$\iint (\nabla \times F)\,ds = \boxed{0}$

11) $C: x^2 + y^2 = 9, \; z = 2$

$\int_C F\,dr = \iint \text{curl} F\,ds$

$\text{curl} F = \nabla \times F = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)k$

$P = 3y, \; \frac{\partial P}{\partial z} = 0, \; \frac{\partial P}{\partial y} = 3$

$Q = -2x, \; \frac{\partial Q}{\partial x} = -2, \; \frac{\partial Q}{\partial z} = 0$

$R = 3y, \; \frac{\partial R}{\partial y} = 3, \; \frac{\partial R}{\partial x} = 0$

$\text{curl} F = 3\hat{i} - 5\hat{k}$

$\iint \text{curl} F\,ds = \iint (3\hat{i} - 5\hat{k}) \cdot (0, 0, 1)$

$= \iint -5\,dS = -5 \times \pi \times 3^2 = \boxed{-45\pi}$

13) $\nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$

$= i(0-1) - j(1-0) + k(0-1)$

$= -i - j - k$

$\overrightarrow{PQ} = \langle 3,0,0\rangle, \ \overrightarrow{PR} = \langle 0,3,3\rangle$

$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} i & j & k \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \langle 0,-9,9\rangle$

$0 - 9(y-0) + 9(z-0) = 0$

$y - z = 0$

$x = u, \ y = v, \ z = v$

$r(u,v) = u\hat{i} + v\hat{j} + v\hat{k}$

$r_u(u,v) = \hat{i}, \quad r_v(u,v) = j + k$

$r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0,-1,1\rangle$

$\text{curl} F \cdot (r_u \times r_v) = \langle -1,-1,-1\rangle \cdot \langle 0,-1,1\rangle = 0 + 1 - 1$

$= 0$

$\oint_C F \cdot dr = \iint_D 0 \, dx\,dy = \boxed{0}$