# 17.1 : 1, 3, 5, 7, 9, 13

**1)** Verify Green's Theorem for the line integral $\int_C xy\,dx + y\,dy$ where C is the unit circle, oriented ccw

$P = xy \quad Q = y$

$\frac{\partial P}{\partial y} = x \quad \frac{\partial Q}{\partial x} = 0$

$0 - x = -x$

$\int_0^{2\pi}\int_0^1 -r\cos\theta \; r\,dr\,d\theta$

$-\frac{r^3\cos\theta}{3}\Big|_0^1 = \int_0^{2\pi} -\frac{\cos\theta}{3}\,d\theta$

$-\frac{\sin\theta}{3}\Big|_0^{2\pi} = 0$

**3)** $\int_C y^2dx + x^2dy$, where C is the boundary of the unit square $0 \le x \le 1, 0 \le y \le 1$

$P = y^2 \quad Q = x^2$

$\frac{\partial P}{\partial y} = 2y \quad \frac{\partial Q}{\partial x} = 2x$

$2x - 2y$
$= 2(x-y)$

$2\int_0^1\int_0^1 x - y\;dx\,dy$

$\frac{x^2}{2} - xy\Big|_0^1 = (\frac{1}{2} - y) - (0) = \frac{1}{2} - y$

$2\int_0^1 \frac{1}{2} - y\,dy = \frac{1}{2}y - \frac{y^2}{2}\Big|_0^1$

$= (\frac{1}{2} - \frac{1}{2})2 = 0$

**5)** $\int_C x^2y\,dx$, where C is the unit circle centered at the origin

$P = x^2y \quad Q = 0$

$\frac{\partial P}{\partial y} = x^2 \quad \frac{\partial Q}{\partial x} = 0$

$\downarrow$

$r^2\cos^2\theta$

$\int_0^{2\pi}\int_0^1 r^2\cos^2\theta\; r\,dr\,d\theta$

$\frac{r^4}{4}\cos^2\theta\Big|_0^1 = \frac{1}{4}\cos^2\theta$

$\frac{1}{4}\int_0^{2\pi}\cos^2\theta\,d\theta \rightarrow \frac{1}{4}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta$

$\frac{1}{4}\left(\frac{\theta + \sin\theta}{2}\Big|_0^{2\pi}\right) = \frac{\pi}{4}$

**7)** $\int_C F\cdot dr$, where $F(x,y) = \langle x^2, x^2\rangle$ and C consists of the arcs $y = x^2$ and $y = x$ for $0 \le x \le 1$

$F\cdot dr = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$

$= \iint_D (2x - 0)\,dA = \iint_D 2x\,dA$

$\int_0^1\int_{x^2}^{x} 2x\,dy\,dx \qquad 2xy\Big|_{x^2}^{x} = 2x^2 - 2x^3$

$\int_0^1 2x^2 - 2x^3\,dx \rightarrow \frac{2}{3}x^3 - \frac{1}{2}x^4\Big|_0^1$

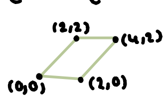
$= \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$

**9)** The line integral of $F(x,y) = \langle e^{x+y}, e^{x-y}\rangle$ along the curve (oriented clockwise) consisting of the line segments by joining the points (0,0), (2,2), (4,2), (2,0), and back to (0,0)

$P = e^{x+y} \quad Q = e^{x-y}$

$\frac{\partial P}{\partial y} = e^{x+y} \quad \frac{\partial Q}{\partial x} = e^{x-y}$

$e^{x-y} - e^{x+y}$




$-\int_0^2\int_0^x e^{x-y} - e^{x+y}\,dy\,dx \;-\; \int_2^4\int_{x-2}^{2} e^{x-y} - e^{x+y}\,dy\,dx$

$-e^{x-y} - e^{x+y}\Big|_0^x \qquad -e^{x-y} - e^{x+y}\Big|_{x-1}^{2}$

$-e^0 - e^{2x} + e^x + e^x \qquad -e^{x-2} - e^{x+2} + e^1 + e^{2x-1}$

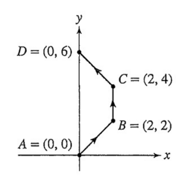
$\int_0^2 -1 - e^{2x} + 2e^x\,dx \qquad \int_2^4 -2e^{x-2} + e + e^{2x-1}\,dx$

$-x - \frac{e^{2x}}{2} + 2e^x\Big|_0^2 \qquad -2e^{x-2} + ex + \frac{e^{2x-1}}{2}\Big|_2^4$

$-(-2 - \frac{e^4}{2} + 2e^2 + \frac{1}{2} - 2) - (-2e^2 + 4e + \frac{e^7}{2} + 2 - 2e - \frac{e^3}{2})$

$\frac{5}{2} - \frac{e^4}{2} + \frac{e^6}{2} - \frac{5e^2}{2}$

**13)** Evaluate $I = \int_C (\sin x + y)\,dx + (3x + y)\,dy$ for the nonclosed path ABCD




$P = \sin x + y \quad Q = 3x + y$

$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 3$

$3 - 1 = 2 \rightarrow 2(\text{Area}) = 2\left(\frac{2+6}{2}\right)(2) = 16$

$r = (1-t)(0,6) + t(0,0) = \langle 0, 6-6t\rangle \quad 0 \le t \le 1$

$x = 0 \quad y = 6 - 6t$
$dx = 0 \quad dy = -6\,dt$

$\int_0^1 (\sin(0) + 6 - 6t)(0) + (3(0) + 6 - 6t)(-6)$

$\int_0^1 -36 + 36t \rightarrow -36t + 18t^2\Big|_0^1$
$= -36 + 18 = -18$

$16 - (-18) = 34$

# 17.2 : 1, 3, 5, 9, 11, 13

**1)** $F = \langle 2xy, x, y+z \rangle$, the surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \le 1$

$r(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le 2\pi$

$r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$F(r(t)) = \langle 2\sin t\cos t, \cos t, \sin t \rangle$

$\langle 2\sin t\cos t, \cos t, \sin t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle$

$= -2\sin^2 t\cos t + \cos^2 t$

$\int_0^{2\pi} (-2\sin^2 t\cos t + \cos^2 t)\, dt$

$-\frac{2}{3}\sin^3 t + \frac{1}{2}t + \frac{1}{4}\sin 2t \Big|_0^{2\pi} = \pi$

$G(r,\theta) = (r\cos\theta, r\sin\theta, 1 - r^2) \quad 0 \le r \le 1, 0 \le \theta \le \pi$

$G_r = (\cos\theta, \sin\theta, -2r)$

$G_\theta = (-r\sin\theta, r\cos\theta, 0)$ $\times$ $= \langle -2r^2\cos\theta, 2r^2\sin\theta, r \rangle$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x & y+z \end{vmatrix} = \langle 1, 0, 1-2x \rangle = \langle 1, 0, 1 - 2r\cos\theta \rangle$$

$\langle 1, 0, 1 - 2r\cos\theta \rangle \cdot \langle 2r^2\cos\theta, 2r^2\sin\theta, r \rangle = r$

$\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \pi$

**3)** $F = \langle e^{y-z}, 0, 0 \rangle$, the square with vertices $(1,0,1)$, $(1,1,1)$, $(0,1,1)$, and $(0,0,1)$

$z = 1$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{y-z} & 0 & 0 \end{vmatrix} = i\left(\frac{\partial}{\partial y}0 - \frac{\partial}{\partial z}0\right) - j\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial z}e^{y-z}\right) + k\left(\frac{\partial}{\partial x}0 - \frac{\partial}{\partial y}e^{y-z}\right)$$

$\langle 0, -e^{y-z}, -e^{y-z} \rangle$

$\int_0^1\int_0^1 \underbrace{\langle 0, -e^{y-z}, -e^{y-z} \rangle \cdot \langle 0, 0, 1 \rangle}_{-e^{y-z}} = e^{-1} - 1$

**5)** $F = \langle e^{z^2} - y, e^{z^3} + x, \cos(xz) \rangle$, the upper hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$ with outward-pointing normal

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^{z^2} - y & e^{z^3} + x & \cos(xz) \end{vmatrix} = i\left(\frac{\partial}{\partial y}\cos(xz) - \frac{\partial}{\partial z}e^{z^3} + x\right) - j\left(\frac{\partial}{\partial x}\cos(xz) - \frac{\partial}{\partial z}e^{z^2} - y\right) + k\left(\frac{\partial}{\partial x}e^{z^3} + x - \frac{\partial}{\partial y}e^{z^2} - y\right)$$

$0 - 3z^2e^{z^3} \qquad -z\sin(xz) - 2ze^{z^2} \qquad 1 + 1$

$\langle -3z^2e^{z^3}, z\sin(xz) + 2ze^{z^2}, 2 \rangle$

$r(t) = \langle \cos t, \sin t, 0 \rangle \quad 0 \le t \le 2\pi$

$r'(t) = \langle -\sin t, \cos t, 0 \rangle$

$F(r(t)) = \langle 1 - \sin t, 1 + \cos t, 1 \rangle$

$\langle 1 - \sin t, 1 + \cos t, 0 \rangle \cdot \langle -\sin t, \cos t, 0 \rangle = 2\pi$

**9)** $F = \langle yz, xz, xy \rangle$, that part of the cylinder $x^2 + y^2 = 1$ that lies between the two planes $z = 1$ and $z = 4$ with outward-pointing unit normal vector

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = i\left(\frac{\partial}{\partial y}xy - \frac{\partial}{\partial z}xz\right) - j\left(\frac{\partial}{\partial x}xy - \frac{\partial}{\partial z}yz\right) + k\left(\frac{\partial}{\partial x}xz - \frac{\partial}{\partial y}yz\right)$$

$= \langle 0, 0, 0 \rangle \rightarrow$ conservative $\rightarrow 0$

11) $F = \langle 3y, -2x, 3y \rangle$, C is the circle $x^2+y^2=9$, $z=2$, oriented counterclockwise as viewed from above

$r = 3$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -2x & 3y \end{vmatrix} = i\left(\frac{\partial}{\partial y}3y + \frac{\partial}{\partial z}2x\right) - j\left(\frac{\partial}{\partial x}3y - \frac{\partial}{\partial z}3y\right) + k\left(\frac{\partial}{\partial x}-2x - \frac{\partial}{\partial y}3y\right)$$

$3+0 \qquad 0-0 \qquad -2-3$

$\langle 3, 0, -5 \rangle$

$$-9\int_0^{2\pi} 3\sin^2 t + 2\cos^2 t \, dt$$

$r(t) = \langle 3\cos t, 3\sin t, 2 \rangle$

$r'(t) = \langle -3\sin t, 3\cos t, 0 \rangle$

$F(r(t)) = \langle 9\sin t, -6\cos t, 9\sin t \rangle$

$\langle 9\sin t, -6\cos t, 9\sin t \rangle \cdot \langle -3\sin t, 3\cos t, 0 \rangle$

$= -27\sin^2 t - 18\cos^2 t$

$-9\left(\frac{5}{2}t - \frac{1}{4}\sin(2t)\Big|_0^{2\pi}\right)$

$-9(5\pi) = -45\pi$

13) $F = \langle y, z, x \rangle$, C is the triangle with vertices $(0,0,0)$, $(3,0,0)$, and $(0,3,3)$, oriented counterclockwise as viewed from above.

$D: \{(x,y) \mid 0 \le x \le 3, 0 \le y \le 3\}$ $\quad g(x,y) = x+y$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = i\left(\frac{\partial}{\partial y}x - \frac{\partial}{\partial z}z\right) - j\left(\frac{\partial}{\partial x}x - \frac{\partial}{\partial z}y\right) + k\left(\frac{\partial}{\partial x}z - \frac{\partial}{\partial y}y\right)$$

$0-1 \qquad 1-0 \qquad 0-1$

$= \langle -1, -1, -1 \rangle$

$$\iint (1(1)) - (-1(1)) - 1 \, dA = \int_0^3\int_0^3 1 \, dy\, dx = 0$$