17.1, 17.2 HW

12/6/20

17.1 : 1, 3, 5, 7, 9, 13

1. Verify Green's Thm. for the line integral $\oint_C xy\,dx + y\,dy$, where C is the unit circle, oriented counterclockwise.

S1. Evaluate line integral

$\gamma(\theta) = \langle \cos\theta, \sin\theta \rangle$, $0 \le \theta \le 2\pi \Rightarrow dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

$xy\,dx + y\,dy = \cos\theta \sin\theta(-\sin\theta\, d\theta) + \sin\theta\cos\theta\, d\theta = (-\cos\theta\sin^2\theta + \sin\theta\cos\theta)\,d\theta$

$$\int_C xy\,dx + y\,dy = \int_0^{2\pi} (-\cos\theta\sin^2\theta + \sin\theta\cos\theta)\,d\theta$$

$$= \int_0^{2\pi} -\cos\theta\sin^2\theta\, d\theta + \int_0^{2\pi} \sin\theta\cos\theta\, d\theta = -\frac{\sin^3\theta}{3}\Big|_0^{2\pi} - \frac{\cos 2\theta}{4}\Big|_0^{2\pi} = 0$$

S2. Evaluate double integral

$P = xy$ & $Q = y \Rightarrow dQ/dx - dP/dy = 0 - x = -x$

$$\iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dx\,dy = \iint_D -x\,dx\,dy = -\iint_D x\,dx\,dy = 0$$

S3. Compare S1. & S2.

Both integrals are equal, as stated in Green's Thm.

3. $\oint_C y^2dx + x^2dy$, where C is the boundary of the □ $\Rightarrow 0 \le x \le 1$, $0 \le y \le 1$

$P = y^2$ & $Q = x^2 \Rightarrow dQ/dx - dP/dy = 2x - 2y$

$\int_C y^2dx + x^2dy = \iint_D dQ/dx - dP/dy\; dA = \iint_D (2x-2y)\,dx\,dy$

$$= \int_0^1\int_0^1 (2x-2y)\,dx\,dy = \int_0^1 (x^2 - 2xy)\Big|_0^1 dy = \int_0^1 (1-2y)\,dy = (y-y^2)\Big|_0^1 = 0$$

5. $\oint_C x^2y\,dx$, where C is the unit ○ centered at the origin

$P = x^2y$ & $Q = 0 \Rightarrow dQ/dx - dP/dy = 0 - x^2 = -x^2$

$I = \int_C x^2y\,dx = \iint_D (dQ/dx - dP/dy)\,dA = \iint_D -x^2\,dA$

$$I = \int_0^{2\pi}\int_0^1 -r^2\cos^2\theta \cdot r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 -r^3\cos^2\theta\,dr\,d\theta = \left(\int_0^{2\pi}\cos^2\theta\,d\theta\right)\left(\int_0^1 -r^3dr\right)$$

$$= \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\Big|_{\theta=0}^{2\pi}\right)\left(-\frac{r^4}{4}\Big|_{r=0}^{1}\right) = \pi \cdot \left(-\frac{1}{4}\right) = -\frac{\pi}{4}$$

7. $\oint_C F \cdot dr$, where $F(x,y) = \langle x^2, y^2 \rangle$ & C consists of the arcs $y = x^2$ & $y = x$ for $0 \le x \le 1$

$I = \int_C F \cdot dr = \iint_D (dQ/dx - dP/dy)\,dA$

$P = Q = x^2 \Rightarrow dQ/dx - dP/dy = 2x - 0 = 2x$

$$I = \iint_D 2x\,dA = \int_0^1 \int_{x^2}^{x} 2x\,dy\,dx = \int_0^1 2xy \Big|_{y=x^2}^{x} dx = \int_0^1 2x(x-x^2)\,dx$$

$$= \int_0^1 (2x^2-2x^3)\,dx = \frac{2}{3}x^3 - \frac{1}{2}x^4 \Big|_0^1 = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$$

9. Compute curl of $F = \langle e^{x+y}, e^{x-y} \rangle$. Here, $P = e^{x+y}$ & $Q = e^{x-y}$

$\Rightarrow dQ/dx - dP/dy = e^{x-y} - e^{x+y} = e^x(e^{-y} - e^y)$

$$\int_C F \cdot dr = \iint_D e^x(e^{-y}-e^y)\,dx\,dy = \int_0^2 \int_y^{y+2} e^x(e^{-y}-e^y)\,dx\,dy = \int_0^2 e^x(e^{-y}-e^y)\Big|_{x=y}^{y+2} dy$$

$$= \int_0^2 (e^{y+2}-e^y)(e^{-y}-e^y)\,dy = \int_0^2 (e^2-1)(1-e^{2y})\,dy = (e^2-1)\left(y - \frac{e^{2y}}{2}\right)\Big|_{y=0}^{2}$$

$$= (e^2-1)\left(2 - \frac{e^4}{2} - \left(-\frac{1}{2}\right)\right) = \frac{(e^2-1)(5-e^4)}{2}$$

13. Evaluate $I = \int_C (\sin x + y)\,dx + (3x+y)\,dy$ for the nonclosed path ABCD in Fig. 20. Use the method of Exercise 12.

Let $F = \langle \sin x + y, 3x+y \rangle \Rightarrow P = \sin x + y$ & $Q = 3x+y$

$\int_{C_1} P\,dx + Q\,dy = \iint_D (dQ/dx - dP/dy)\,dA = \iint_D (3-1)\,dA = 2\iint_D dA = 2\,\text{Area}(D)$

$\text{Area}(D) = ((\overline{BC} + \overline{AD})h)/2 = ((2+6)\cdot 2)/2 = 8$

$\int_{C_1} P\,dx + Q\,dy = 2 \cdot 8 = 16$

$\int_C P\,dx + Q\,dy = \int_{\overline{DA}} P\,dx + Q\,dy = 16$, $\overline{DA}: x=0,\ y=t$, t varies from 6 to 0

$$\int_{\overline{DA}} P\,dx + Q\,dy = \int_6^0 F(0,t) \cdot \frac{d}{dt}\langle 0,t\rangle\,dt = \int_6^0 \langle \sin 0 + t, 3\cdot 0 + t\rangle \cdot \langle 0,1\rangle\,dt$$

$$= \int_6^0 \langle t,t\rangle \cdot \langle 0,1\rangle\,dt = \int_6^0 t\,dt = \frac{t^2}{2}\Big|_{t=6}^{0} = -18$$

$\int_C P\,dx + Q\,dy - 18 = 16$ or $\int_C P\,dx + Q\,dy = 34$

17.2 : # 1, 3, 5, 9, 11, 13

1. $F = \langle 2xy, x, y+z \rangle$, the surface $z = 1 - x^2 - y^2$ for $x^2+y^2 \leq 1$

$\int_C F \cdot dr = \iint_S \text{curl}(F) \cdot ds$

S1. Compute line integral around boundary curve

$\gamma(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \leq t \leq 2\pi$

$F(\gamma(t)) = \langle 2\cos t \sin t, \cos t, \sin t \rangle$

$\gamma'(t) = \langle -\sin t, \cos t, 0 \rangle$

$F(r(t)) \cdot r'(t) = \langle 2\cos t \sin t, \cos t, \sin t\rangle \cdot \langle -\sin t, \cos t, 0\rangle = -2\cos t \sin^2 t + \cos^2 t$

$\int_C F dr = \int_0^{2\pi} (-2\cos t \sin^2 t + \cos^2 t)\, dt = -\frac{2\sin^3 t}{3} + \frac{t}{2} + \frac{\sin 2t}{4} \Big|_0^{2\pi} = \pi$

S2. Compute flux of curl through surface

$\Phi(\theta, t) = (t\cos\theta, t\sin\theta, 1-t^2), \quad 0 \le t \le 1, \quad 0 \le \theta \le 2\pi$

$T_\theta = d\Phi/d\theta = \langle -t\sin\theta, t\cos\theta, 0\rangle$

$T_t = d\Phi/dt = \langle \cos\theta, \sin\theta, -2t\rangle$

$$T_\theta \times T_t = \begin{vmatrix} i & j & K \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & -2t \end{vmatrix} = (-2t^2\cos\theta)i - (2t^2\sin\theta)j - t(\sin^2\theta + \cos^2\theta)K$$
$$= (-2t^2\cos\theta)i - (2t^2\sin\theta)j - tK$$

$n = \langle 2t^2\cos\theta, 2t^2\sin\theta, t\rangle$

$$\text{curl}(F) = \begin{vmatrix} i & j & K \\ d/dx & d/dy & d/dz \\ 2xy & x & y+z \end{vmatrix} = i + (1-2x)K = \langle 1, 0, 1-2x\rangle$$
$$= \langle 1, 0, 1-2t\cos\theta\rangle$$

$\text{curl}(F) \cdot n = \langle 1, 0, 1-2t\cos\theta\rangle \cdot \langle 2t^2\cos\theta, 2t^2\sin\theta, t\rangle = 2t^2\cos\theta + t - 2t^2\cos\theta = t$

$\iint_S \text{curl}(F) \cdot dS = \int_0^{2\pi}\int_0^1 t\, dt\, d\theta = 2\pi \int_0^1 t\, dt = 2\pi \cdot \frac{t^2}{2}\Big|_0^1 = \pi$

3. $F = \langle e^{y-z}, 0, 0\rangle$, the $\square$ w/ vertices $(1,0,1)$, $(1,1,1)$, $(0,1,1)$, & $(0,0,1)$

S1. Compute integral around the boundary curve

$\int_C F \cdot dr = \sum_{i=1}^{4} \int_{C_i} F \cdot dr = \int_0^1 e^{-1} dt + 0 + \int_0^1 (-1) dt + 0 = e^{-1} - 1$

S2. Compute curl.

$$\text{curl}(F) = \begin{vmatrix} i & j & K \\ d/dx & d/dy & d/dz \\ e^{y-z} & 0 & 0 \end{vmatrix} = -e^{y-z} j - e^{y-z} K = \langle 0, -e^{y-z}, -e^{y-z}\rangle$$

S3. Compute Flux of curl through surface

$\iint_S \text{curl}(F) \cdot ds = \iint_D -e^{y-1} dA = \int_0^1\int_0^1 -e^{y-1} dy\, dx = \int_0^1 -e^{y-1} dy = e^{-1} - 1$

5. $F = \langle e^{z^2} - y, e^{z^3} + x, \cos(xz)\rangle$

$$\text{curl}(F) = \begin{vmatrix} i & j & K \\ dx & dy & dz \\ e^{z^2}-y & e^{z^3}+x & \cos(xz) \end{vmatrix} = \langle -3z^2 e^{z^3}, 2ze^{z^2} + z\sin(xz), 2\rangle$$

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$$\int_C F \cdot dS = \int_0^{2\pi} (1 - \sin t + \cos t)\,dt = t + \cos t + \sin t \Big|_0^{2\pi} = 2\pi$$

9. $\mathrm{curl}(F) = \begin{vmatrix} i & j & K \\ dx & dy & dz \\ yz & xz & xy \end{vmatrix} = \langle 0,0,0 \rangle$

$$\int_{C_1} F \cdot ds + \int_{C_4} F \cdot ds = \int_0^{2\pi} \cos 2t\,dt + \int_0^{2\pi} (-4\cos 2t)\,dt = -3\int_0^{2\pi} \cos 2t\,dt = 0$$

11. $F = \langle 3y, -2x, 3y \rangle$

$r(r,\theta) = \langle r\cos\theta, r\sin\theta, 2 \rangle, \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi$

$\mathrm{curl}(F) = \begin{vmatrix} i & j & K \\ dx & dy & dz \\ 3y & -2x & 3y \end{vmatrix} = \langle 3, 0, -5 \rangle$

$r_r = \langle \cos\theta, \sin\theta, 0 \rangle$

$r_\theta = \langle -r\sin\theta, r\cos\theta, 0 \rangle$ $\Rightarrow$ $r_r \times r_\theta = \begin{vmatrix} i & j & K \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0,0,r \rangle$

$$\iint_S \mathrm{curl}(F) \cdot dr = \int_0^3 \int_0^{2\pi} \langle 3,0,-5 \rangle \cdot \langle 0,0,r \rangle\,d\theta\,dr = \int_0^3 \int_0^{2\pi} -5r\,d\theta\,dr$$

$$= \int_0^3 (-10\pi r)\,dr = -5\pi r^2 \Big|_0^3 = -45\pi$$

13. $F = \langle y, z, x \rangle$ $\Rightarrow$ $r(x,y) = \langle x, y, y \rangle, \quad 0 \le x \le 3, \quad 0 \le y \le 3-x$

$\begin{vmatrix} i & j & K \\ 3 & 0 & 0 \\ 0 & 3 & 3 \end{vmatrix} = \langle 0, -9, 9 \rangle$

$\mathrm{curl}(F) = \begin{vmatrix} i & j & K \\ dx & dy & dz \\ y & z & x \end{vmatrix} = \langle -1,-1,-1 \rangle$

$r_x = \langle 1, 0, 0 \rangle$

$r_y = \langle 0, 1, 1 \rangle$ $\Rightarrow$ $r_x \times r_y = \begin{vmatrix} i & j & K \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \langle 0, -1, 1 \rangle$

$$\iint_S \mathrm{curl}(F) \cdot dr = \int_0^3 \int_0^{3-x} \langle -1,-1,-1 \rangle \cdot \langle 0,-1,1 \rangle\,dy\,dx$$

$$= \int_0^3 \int_0^{3-x} 0\,dy\,dx = 0$$