HW due 12/13

sec 17.3: 1, 3, 5, 7, 11, 15

sec 17.3

1. $S_1 : \phi_1(x,z) = (x, 0, z)$ $\quad 0 \le x \le 4 \ + \ 0 \le z \le 3$

$n = \langle 0, -1, 0 \rangle$

$S_2 : \phi_2(y,z) = (0, y, z)$ $\quad 0 \le y \le 2 \ + \ 0 \le z \le 3$

$n = \langle -1, 0, 0 \rangle$

$S_3 : \phi_3(x,z) = (x, 2, z)$ $\quad 0 \le x \le 4 \ + \ 0 \le z \le 3$

$n = \langle 0, 1, 0 \rangle$

$S_4 : \phi_4(y,z) = (4, y, z)$ $\quad 0 \le y \le 2 \ + \ 0 \le z \le 3$

$n = \langle 1, 0, 0 \rangle$

$S_5 : \phi_5(xy) = (x, y, 0)$ $\quad 0 \le x \le 4 \ + \ 0 \le y \le 2$

$n = \langle 0, 0, -1 \rangle$

$S_6 : \phi_6(xy) = (x, y, 3)$ $\quad 0 \le x \le 4 \ + \ 0 \le y \le 2$

$n = \langle 0, 0, 1 \rangle$

$\int_0^3\int_0^4 -x\,dx\,dz = \int_0^3 \left(-\frac{x^2}{2}\right)\Big|_0^4 dz = \int_0^3 -8\,dz = -24$

$\int_0^3\int_0^2 -z\,dy\,dz = \int_0^3 -zy\Big|_0^2 dz = \int_0^3 -2z\,dz = -9$

$\int_0^3\int_0^4 x\,dx\,dz = \int_0^3 \left(\frac{x^2}{2}\right)\Big|_0^4 dz = \int_0^3 8\,dz = 24$

$\int_0^3\int_0^2 z\,dy\,dz = \int_0^3 zy\Big|_0^2 dz = \int_0^3 2z\,dz = 9$

$\int_0^2\int_0^4 -y\,dx\,dy = \int_0^2 -xy\Big|_0^4 dy = \int_0^2 -4y\,dy = -8$

$\int_0^2\int_0^4 y\,dx\,dy = \int_0^2 xy\Big|_0^4 dy = \int_0^2 4y\,dy = 8$

$\rightarrow (-24) + (-9) + 24 + 9 + (-8) + 8 = \boxed{0}$

3. $\iint_S F\,ds = \iint_{S_1} F\,ds + \iint_{S_2} F\cdot ds + \iint_{S_3} F\cdot ds$

$G(\theta, z) = (\cos\theta, \sin\theta, z)$ $\quad 0 \le \theta \le 2\pi \quad 0 \le z \le 2$

$N = T_\theta \times T_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$

$F(G(\theta,z))\cdot N = \langle z\cos\theta, 3z, 3\sin\theta \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle$

$= z\cos^2\theta + 3z\sin\theta$

$\int_0^2\int_0^{2\pi} z\cos^2\theta + 3z\sin\theta\,d\theta\,dz$

$= \int_0^2 \left(\frac{\sin 2\theta}{2} + \theta - 3z\cos\theta\right)\Big|_0^{2\pi} dz$

$= \int_0^2 2\pi\,dz = \underline{4\pi}$

$G(r,\theta) = \langle r\cos\theta, r\sin\theta, 2 \rangle$ $\quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$

$T_r \times T_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = \langle 0, 0, r \rangle$

$N = \langle 0, 0, 1 \rangle$

$F(G(r,\theta))\cdot N = \langle 2r\cos\theta, 1, 3r\sin\theta \rangle \cdot \langle 0, 0, 1 \rangle$

$= 3r\sin\theta$

$\int_0^1\int_0^{2\pi} 3r\sin\theta\,d\theta\,dr = \int_0^1 -3r\cos\theta\Big|_0^{2\pi} dr = \int_0^1 0\,dr = \underline{0}$

$\iint_S F\,ds = 4\pi + 0 + 0 = \boxed{4\pi}$

5. $F(x,y,z) = \langle 0, 0, z^3/3 \rangle$ $\quad S: x^2 + y^2 + z^2 = 1$

$\operatorname{div} F = \frac{d}{dx}(0) + \frac{d}{dy}(0) + \frac{d}{dz}\left(\frac{z^3}{3}\right) = z^2$

$x = \rho\cos\theta\sin\phi \quad y = \rho\sin\theta\sin\phi \quad z = \rho\cos\phi$

$0 \le \theta \le 2\pi \quad 0 \le \phi \le \pi \quad 0 \le \rho \le 1$

$= \int_0^\pi\int_0^{2\pi}\int_0^1 (\rho\cos\phi)^2 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi$

$= \int_0^\pi\int_0^{2\pi} \frac{1}{5}\rho^5\cos^2\phi\Big|_0^1 d\theta\,d\phi$

$= \int_0^\pi \frac{1}{5}\cos^2\phi\,\theta\Big|_0^{2\pi} d\phi = \int_0^\pi \frac{2\pi}{5}\cos^2\phi\,d\phi$

$= \left(\frac{2\pi}{5}\right)\left(-\frac{\cos^3\phi}{3}\right)\Big|_0^\pi = \boxed{\frac{4\pi}{15}}$

7. $F(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ $\quad S: x^2 + y^2 \le 4 \quad 0 \le z \le 3$

$\operatorname{div} F = \frac{d}{dx}(xy^2) + \frac{d}{dy}(yz^2) + \frac{d}{dz}(zx^2) = x^2 + y^2 + z^2$

$\int_0^\pi\int_0^{2\pi}\int_0^2 r^2\,dr\,d\theta\,d\phi = \int_0^\pi\int_0^{2\pi} \frac{r^3}{3}\Big|_0^2 d\theta\,d\phi$

$= \int_0^\pi \frac{8}{3}\theta\Big|_0^{2\pi} d\phi = \frac{16\pi}{3}\phi\Big|_0^\pi = \boxed{\frac{16\pi^2}{3}}$

11. $F(x,y,z) = \langle x^3, 0, z^3 \rangle$ $\quad S: x^2 + y^2 + z^2 \le 4 \quad x \ge 0 \quad y \ge 0 \quad z \ge 0$

$\operatorname{div} F = \frac{d}{dx}(x^3) + \frac{d}{dy}(0) + \frac{d}{dz}(z^3) = 3x^2 + 3z^2$

$\int_0^\pi\int_0^{2\pi}\int_0^2 (\rho\cos\theta\sin\phi)^2 + (\rho\cos\theta)^2\,d\rho\,d\theta\,d\phi$

$= \int_0^\pi\int_0^{2\pi} \frac{8}{3}\cos^2\theta\sin^2\phi + \frac{8}{3}\cos^2\theta\,d\theta\,d\phi$

$= \int_0^\pi -\frac{8}{9}\cos^3\theta\sin^2\phi - \frac{8}{9}\cos^3\theta\Big|_0^{2\pi} d\phi = \int_0^\pi 0\,d\phi = \boxed{0}$

15. $F = \langle x+y, z, z-x \rangle$ $S: z = 9 - x^2 - y^2$ + xy-plane

$\text{div}(F) = \frac{d}{dx}(x+y) + \frac{d}{dy}(z) + \frac{d}{dz}(z-x) = 2$

$$\int_0^{2\pi}\int_0^3\int_0^{9-r^2} 2r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^3 2rz \Big|_0^{9-r^2} dr \, d\theta$$

$$= \int_0^{2\pi}\int_0^3 18r - 2r^3 \, dr \, d\theta = \int_0^{2\pi} 9r^2 - \frac{r^4}{2} \Big|_0^3 d\theta$$

$$= \int_0^{2\pi} 81 - \frac{81}{2} \, d\theta = \frac{81}{2}\theta \Big|_0^{2\pi} = \boxed{81\pi}$$