17.3 (Dec 13th)

17.3: # 1, 3, 5, 7, 11, 15

1) six faces separate

$= \int_0^2 \int_0^4 (z,x,y)\cdot(0,0,1)\,dx\,dy + \int_0^3 \int_0^2 (z,x,y)\cdot(1,0,0)\,dy\,dz$
$+ \int_0^4 \int_0^3 (z,x,y)\cdot(0,1,0)\,dz\,dx + \int_0^3 \int_0^4 (z,x,y)\cdot(0,0,-1)\,dy\,dz$
$+ \int_0^3 \int_0^2 (z,x,y)\cdot(-1,0,0)\,dy\,dz + \int_0^4 \int_0^3 (z,x,y)\cdot(0,-1,0)\,dz\,dx$
$= \iint_{S_1} y\,dx\,dy + \iint_{S_2} z\,dy\,dz + \iint_{S_3} x\,dz\,dx + \iint_{S} (-z)\,dy\,dz$
$+ \iint_{S} (-y)\,dx\,dy + \iint_{S_7} (-x)\,dz\,dx$

$\iint_{S_1} y\,dy\,dx = \int_0^4 \int_0^2 y\,dy\,dx$
$\iint_{S_5} -(y)\,dy\,dx = \int_0^4 \int_0^2 -y\,dy\,dx$

$div(f) = \nabla(z,x,y) = 0$
$\therefore \iiint_W div(F)\,dV = 0$

3) $F(x,y,z) = (2x, 3z, 3y)$

$r_u = (-\sin u, \cos u, 0)$

$r_v = (0,0,1)$

$$\begin{vmatrix} i & j & k \\ -\sin u & \cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} \to i(\cos u - 0) - j(-\sin u - 0) + k(0-0)$$
$\to (\cos u, \sin u, 0)$

$\iint F\cdot N\,dS = \int_0^2 \int_0^{2\pi} (2\cos u, 3v, 3\sin u)\cdot(\cos u, \sin u, 0)\,du\,dv$
$= \int_0^2 \int_0^{2\pi} (2\cos^2 u + 3v\sin u + (3\sin u)(0))\,du\,dv$
$= \int_0^2 \int_0^{2\pi} (2\cos^2 u + 3v\sin u)\,du\,dv$
$= \int_0^2 (2\pi)\,dv \to 4\pi$

$$\int_0^{2\pi}\int_0^1 (2r\cos\theta, 6, 3r\sin\theta)\cdot(0,0,1)\,r\,d\theta\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (3r^2\sin\theta)\,dr\,d\theta$$

$$= \int_0^{2\pi}(\sin\theta)\,d\theta \to 0$$

$$\int_0^{2\pi}\int_0^1 (3r^2\sin\theta)\,dr\,d\theta = 0$$

$\operatorname{div} f = 2$

$$\int_0^{2\pi}\int_0^1\int_0^2 2r\,dr\,d\theta \to \int_0^{2\pi} 2\,d\theta$$

$$2(2\pi) = \boxed{4\pi}$$

5) $F \cdot\cdot\ \langle 0, 0, \frac{z^3}{3}\rangle$

$\operatorname{div} F = 0 + 0 + z^2 = z^2$

$$\iiint_W \operatorname{div}(F)\,dV = \iiint_W z^2\,dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^1 \rho^2\cos^2\phi \cdot \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= 2\pi\left(-\frac{\cos^3\phi}{3}\Big|_0^{\pi}\right)\left(\frac{\rho^5}{5}\Big|_0^1\right)$$

$$= (2\pi)\left(-\frac{1}{3}\right)(-1-1)\left(\frac{1}{5}\right)$$

$$= \boxed{\frac{4\pi}{15}}$$

11) $\operatorname{div} f = 3x^2 + 3z^2$

$$= \iiint_V \operatorname{div} f\,dV = 3x^2 + 3z^2\,dx\,dy\,dz$$

Let $x = r\cos\theta\sin\phi$

$z = r\cos\phi$, $y = r\sin\phi\sin\theta$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 3(r^2\sin^2\phi\cos^2\theta + r^2\cos^2\phi)\,r^2\sin\phi\,dr\,d\theta\,d\phi$$

$$= 3 \times \frac{32}{5} \times \left[\frac{3}{4}-(-\cos\phi)\Big|_0^{\pi/2} + \frac{1}{4}\left[\frac{\cos(3\phi)}{3}\Big|_0^{\pi/2}\right]\right]$$

$$\times \left[\frac{1}{2}(\theta)\Big|_0^{\pi/2} + \frac{1}{2}\left(\frac{\sin 2\theta}{2}\Big|_0^{\pi/2}\right)\right] + 3\left(\frac{32}{5}\right)\times\frac{\pi}{2}\times\left(-\frac{1}{3}\right)$$

$$= \frac{32\pi}{10} + \frac{32\pi}{10}$$

$$= \boxed{\frac{32\pi}{5}}$$

15) $\iiint_W \nabla (x+y, z, z^2 - x) dV$

$= \int_0^3 \int_0^{2\pi} \int_0^{4-r^2} 2r \, dz \, d\theta \, dr$

$\int_0^5 4\pi r (9 - r^2) dr$

$= 4\pi (9 \cdot \frac{9}{2} - \frac{81}{4}) = 4\pi \times \frac{81}{4}$

$= \boxed{81\pi}$

7) $\iiint_W \mathrm{div}(F) dV$

$\mathrm{div}(F) = x^2 + y^2 + z^2$

$= \int_0^3 \int_0^{2\pi} \int_0^2 (r\cos\theta)^2 + (r\sin\theta)^2 + z^2 \, r \, dr \, d\theta \, dz$

$= \int_0^3 \int_0^{2\pi} \int_0^2 (r^2 + z^2) r \, dr \, d\theta \, dz = 24\pi + 36\pi$

$\boxed{60\pi}$