17.3 Homework

1. $F(x,y,z) = \langle z, x, y \rangle$

$$\int_C F \cdot dV = \int z\,dx + x\,dy + y\,dz$$

$$\int_0^4 \int_0^2 \int_0^3 (z + x + y)\,dz\,dy\,dx$$

$$= \int_0^3 z + xz + yz = \frac{z^2}{2} + xz + yz \Big|_0^1 = \frac{1}{2} + x + y$$

$$= \int_0^2 \frac{1}{2} + x + y\,dy = \frac{y}{2} + xy + \frac{y^2}{2} \Big|_0^2 =$$

$$1 + 2x + 2 = 3 + 2x$$

$$= \int_0^4 2x + 3\,dx = x^2 + 3x \Big|_0^4 = 16 + 12$$

$$= \boxed{28}$$

3. $F(x,y,z) = \langle 2x, 3z, 3y \rangle \qquad x^2 + y^2 \le 1 \qquad 0 \le z \le 2$

$$\int_0^1 \int_0^1 \int_0^2 2x + 3z + 3y\,dz\,dy\,dx$$

$$= \int_0^2 2x + 3z + 3y\,dz = 2xz + \frac{3z^2}{2} + 3yz \Big|_0^2$$

$$= 4x + 6 + 6y$$

$$\int_0^1 4x + 6y + 6\, dy = 4yx + \frac{6y^2}{2} + 6y \Big|_0^1$$

$$= 4x + 3 + 6 = 9 + 4x$$

$$\int_0^1 9 + 4x\, dx = 9x + \frac{4x^2}{2} \Big|_0^1 = 9 + 2 = \boxed{11}$$

5. $F(x, y, z) = \langle 0, 0, z^3/3 \rangle \quad x^2 + y^2 + z^2 = 1$

$$\iint F \cdot dS = \iiint \operatorname{div} F \cdot dV$$

$$\operatorname{div} F = 0 + 0 + \frac{3z^2}{3} = z^2$$

$z = \rho \cos\phi \qquad 0 \le \rho \le 1 \qquad 0 \le \phi \le \pi \qquad 0 \le \theta \le 2\pi$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^1 (\rho \cos\phi)^2 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\int_0^1 \rho^4 \cos^2\phi \sin\phi \, d\rho = \frac{\rho^5}{5} \Big|_0^1 = \frac{1}{5}$$

$$\frac{1}{5} \int_0^{2\pi} \sin\phi \cos^2\phi \, d\theta = \theta \Big|_0^{2\pi} = 2\pi$$

$$\frac{2\pi}{5} \int_0^{\pi} \sin\phi \cos^2\phi \, d\phi = -\frac{\cos^3\phi}{3} \Big|_0^{\pi} = \frac{2}{3}$$

$\boxed{\frac{4\pi}{15}}$

7. $F(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$ $\quad x^2+y^2 \le 4$

$0 \le z \le 3$

$div F = y^2 + z^2 + x^2$

$0 \le r \le 2$

$0 \le z \le 3$

$0 \le \theta \le 2\pi$

$x = r\cos\theta \quad y = r\sin\theta \quad z = z?$

$div F = (x^2+y^2) + z^2 = r^2 + z^2$

$$\int_0^{2\pi}\int_0^{2}\int_0^{3} (r^2+z^2)\, r\, dz\, dr\, d\theta$$

$$\int_0^3 r^3 + rz^2\, dz = r^3 z + \frac{rz^3}{3}\Big|_0^3 = 3r^3 + 9r$$

$$\int_0^2 9r + 3r^3\, dr = 9r^2 + \frac{3r^4}{4}\Big|_0^2 = 36 + 12 = 48$$

$$\int_0^{2\pi} 48\, d\theta = 48\theta\Big|_0^{2\pi} = \boxed{96\pi}$$

11. $F(x,y,z) = \langle x^3, 0, z^3 \rangle$

$x^2+y^2+z^2 \le 4 \qquad x \ge 0 \quad y \ge 0 \quad z \ge 0$

$\operatorname{div} F = 3x^2 + 3z^2$

$x = \rho\cos\theta\sin\phi \qquad y = \rho\sin\theta\sin\phi \qquad z = \rho\cos\phi$

$0 \le \rho \le 2 \qquad 0 \le \theta \le \pi/2 \qquad 0 \le \phi \le \pi/2$

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \left(3(\rho\cos\theta\sin\phi)^2 + 3(\rho\cos\phi)^2\right)\rho^2\sin\phi \, d\rho\, d\theta\, d\phi$$

$\approx 8.4\pi$

15. $F(x,y,z) = \langle x+y, 2, z-x \rangle$

$\operatorname{div} F = 1 + 0 + 1 = 2$

$0 = 9 - x^2 - y^2$

$9 \ge x^2 + y^2$

$0 \le r \le 3$

$0 \le \theta \le 2\pi$

I think we use cylindrical, but not sure what the range of z is.