17.3 HW

17.3: #1, 3, 5, 7, 11, 15

1. $F(x,y,z) = \langle z, x, y \rangle$, the box $[0,4] \times [0,2] \times [0,3]$

$$\iint_S F \cdot ds = \iiint_R \operatorname{div}(F)\,dV = \sum_{i=1}^{6} \iint_{S_i} F \cdot dS$$
$$= -24 - 9 + 24 + 9 - 8 + 8 = 0$$

$$\operatorname{div}(F) = d/dx\,(z) + d/dy\,(x) + d/dz\,(y) = 0$$

$$\iiint_R \operatorname{div}(F)\,dV = \iiint_R 0\,dV = 0$$

3. $F(x,y,z) = \langle 2x, 3z, 3y \rangle$, the region $x^2+y^2 \leq 1$, $0 \leq z \leq 2$

$$\iint_S F \cdot dS = \iiint_R \operatorname{div}(F)\,dV$$

S1. Integral over the side of the cylinder: $\Phi(\theta, z) = (\cos\theta, \sin\theta, z)$, $\begin{matrix} 0 \leq \theta \leq 2\pi \\ 0 \leq z \leq 2 \end{matrix}$

$N = \langle \cos\theta, \sin\theta, 0 \rangle$

$$F(\Phi(\theta,z)) \cdot N = \langle 2\cos\theta, 3z, 3\sin\theta \rangle \cdot \langle \cos\theta, \sin\theta, 0 \rangle = 2\cos^2\theta + 3z\sin\theta$$

$$\iint_{side} F \cdot dS = \int_0^2 \int_0^{2\pi} (2\cos^2\theta + 3z\sin\theta)\,d\theta\,dz = 4 \cdot \left(\frac{\theta}{2} + \frac{\sin 2\theta}{4} \Big|_0^{2\pi} \right) + 0 = 4\pi$$

S2. Integral over the top of the cylinder: $\Phi(x,y) = (x,y,2)$, $D = \{(x,y) : x^2+y^2 \leq 1\}$

$N = T_x \times T_y = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle = i \times j = k = \langle 0,0,1 \rangle$

$F(\Phi(x,y)) \cdot N = \langle 2x, 6, 3y \rangle \cdot \langle 0,0,1 \rangle = 3y$

$$\iint_{Top} F \cdot dS = \iint_D 3y\,dA = 0$$

S3. Integral over bottom of cylinder: $\Phi(x,y) = (x,y,0)$, $(x,y) \in D$

$N = \langle 0,0,-1 \rangle \Rightarrow F(\Phi(x,y)) \cdot N = \langle 2x, 0, 3y \rangle \cdot \langle 0,0,-1 \rangle = -3y$

$$\iint_{bottom} F \cdot dS = \iint_D -3y\,dA = 0$$

$$\iint_S F \cdot dS = \iint_{Side} F \cdot dS + \iint_{top} F \cdot dS + \iint_{bottom} F \cdot dS = 4\pi + 0 + 0 = \underline{4\pi}$$

S4. Compare w/ integral of divergence

$$\operatorname{div}(F) = \operatorname{div}\langle 2x, 3z, 3y \rangle = d/dx\,(2x) + d/dy\,(3z) + d/dz\,(3y) = 2$$

$$\iiint_R \operatorname{div}(F)\,dV = \iiint_R 2\,dV = 2 \iiint_R dV = 2\,\mathrm{Vol}(R) = 2 \cdot \pi \cdot 2 = \underline{4\pi}$$

5. $F(x,y,z) = \langle 0, 0, z^3/3 \rangle$, S is the sphere $x^2+y^2+z^2=1$

$\operatorname{div} F = d/dx\,(0) + d/dy\,(0) + d/dz\,(z^3/3) = z^2$

$$\iint_S F \cdot dS = \iiint_W \operatorname{div}(F)\, dV = \iiint_W z^2\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \cos^2\phi \sin\phi \, d\phi \cdot \int_0^1 \rho^4\, d\rho = (2\pi)\cdot\left(-\frac{\cos^3\phi}{3}\Big|_0^{\pi}\right)\cdot\left(\frac{\rho^5}{5}\Big|_0^1\right)$$

$$= 2\pi\left(-\frac{1}{3}(-1-1)\right)\left(\frac{1}{5}\right) = 2\pi\left(\frac{2}{3}\right)\left(\frac{1}{5}\right) = \frac{4\pi}{15}$$

7. $F(x,y,z) = \langle xy^2, yz^2, zx^2 \rangle$, S is the boundary of the cylinder given by $x^2+y^2 \le 4$, $0 \le z \le 3$

$\operatorname{div} F = d/dx\,(xy^2) + d/dy\,(yz^2) + d/dz\,(zx^2) = x^2+y^2+z^2$

$$\iint_S F \cdot dS = \iiint_W \operatorname{div} F\, dV = \int_0^3\int_0^2\int_0^{2\pi} (r^2+z^2)\, r\, d\theta\, dr\, dz = 2\pi\int_0^3\int_0^2 (r^3 + rz^2)\, dr\, dz$$

$$= 2\pi\int_0^3 \left(\frac{1}{4}r^4 + \frac{1}{2}r^2z^2\right)\Big|_{r=0}^{2} dz = 2\pi\int_0^3 (4+2z^2)\, dz = 2\pi\left(4z + \frac{2}{3}z^3\right)\Big|_0^3 = 60\pi$$

11. $F(x,y,z) = \langle x^3, 0, z^3 \rangle$, S is the boundary of the region in the first octant of space given by $x^2+y^2+z^2 \le 4$, $x \ge 0$, $y \ge 0$, $z \ge 0$

$\operatorname{div}(F) = d/dx\,(x^3) + d/dy\,(0) + d/dz\,(z^3) = 3x^2+3z^2 = 3(x^2+z^2)$

$$\iint_S F \cdot dS = \iiint_W \operatorname{div}(F)\, dV = \iiint_W 3(x^2+z^2)\, dV \; ; \; x^2+z^2 = \rho^2(1-\sin^2\phi\,\sin^2\theta)$$

$$= 3\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^2(1-\sin^2\phi\,\sin^2\theta)\cdot\rho^2\sin\phi\, d\rho\, d\phi\, d\theta = \frac{32}{5}\left(2\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{\pi/2} = \frac{32\pi}{5}$$

15. $F(x,y,z) = \langle x+y, z, z-x \rangle$, S is the boundary of the region between the paraboloid $z = 9-x^2-y^2$ & the xy-plane

$\operatorname{div}(F) = d/dx\,(x+y) + d/dy\,(z) + d/dz\,(z-x) = 1+0+1 = 2$

$$\iint_S F \cdot dS = \iiint_W \operatorname{div}(F)\, dV = \iiint_W 2\, dV = \iint_D \int_0^{9-x^2-y^2} 2\, dz\, dx\, dy = \iint_D 2z\Big|_0^{9-x^2-y^2} dx\, dy$$

$$= \iint_W 2(9-x^2-y^2)\, dx\, dy$$

$x = r\cos\theta$, $y = r\sin\theta$, $0 \le r \le 3$, $0 \le \theta \le 2\pi$

$$\iint_S F \cdot dS = \int_0^{2\pi}\int_0^3 2(9-r^2)\, r\, dr\, d\theta = 4\pi\int_0^3 (9r - r^3)\, dr = 4\pi\left(\frac{9r^2}{2} - \frac{r^4}{4}\Big|_0^3\right) = 81\pi$$