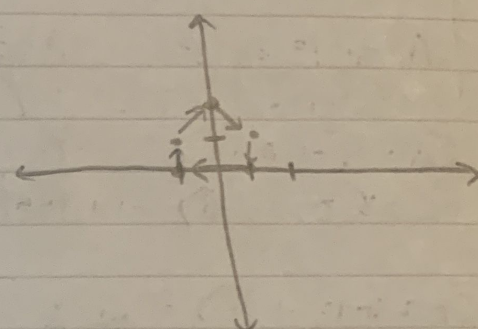
Multivariable Calculus Final Daniel Lim

1. $\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy$

$= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \qquad \frac{\partial Q}{\partial x} = 11 \qquad \frac{\partial P}{\partial y} = 5$

$= \iint_D (11-5)\,dA$

$= 6\int_0^1 \int_0^{2-x} dy\,dx$



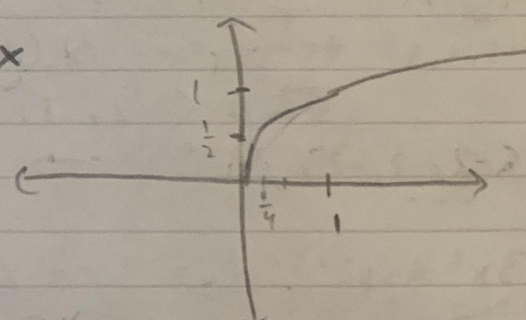
$= 6\int_0^1 2 - x\,dx$

$= 6\left[2x - \frac{x^2}{2}\right]_0^1 \rightarrow 6\left[2 - \frac{1}{2}\right] = 9$

For half of dA so x2 $\rightarrow$ $\boxed{18}$

2. $\int_{\frac{1}{4}}^{1} \int_0^{\sqrt{x}} f(x,y)\,dy\,dx$

$= \boxed{\int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy}$



3. $2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7$

$f_x = -2\sin(x+y) - 4\sin(x+z) \rightarrow f_x(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}) = -3\sqrt{3}$

$f_y = -2\sin(x+y) - 8\sin(y+z) \qquad = -5\sqrt{3}$

$f_z = -4\sin(x+z) - 8\sin(y+z) \qquad = -6\sqrt{3}$

$-3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6}) = 0$

$-3\sqrt{3}x - 5\sqrt{3}y - 6\sqrt{3}z + \frac{7\pi\sqrt{3}}{3} = 0$

$\boxed{z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}}$

5. $A = (0,0,0)$ $B = (1,0,1)$ $C = (1,1,0)$

$\vec{AB} = \langle 1,0,1 \rangle$
$\vec{AC} = \langle 1,1,0 \rangle$
$\frac{\vec{AB} \cdot \vec{AC}}{|\vec{AB}| \cdot |\vec{AC}|} = \frac{1}{2} \Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = 60°$

$\boxed{\text{Angle } A = \frac{\pi}{3} \quad \text{Angle } B = \frac{\pi}{3} \quad \text{Angle } C = \frac{\pi}{3}}$

$\vec{BA} = \langle -1,0,-1 \rangle$
$\vec{BC} = \langle 0,1,-1 \rangle$
$\frac{\vec{BA} \cdot \vec{BC}}{|\vec{BA}| \cdot |\vec{BC}|} = \frac{1}{2} \Rightarrow \cos^{-1}\left(\frac{1}{2}\right) = 60°$

$\vec{CA} = \langle -1,-1,0 \rangle$
$\vec{CB} = \langle 0,-1,1 \rangle$
$= \frac{1}{2} \rightarrow 60°$

6. $f(x,y,z) = x^3 + y^3 + z^3 + xyz$ at $(1,1,1)$

$P = (1,1,1)$ to the point $Q(-1,-1,-1)$
$\vec{PQ} = \langle -2,-2,-2 \rangle$
$\vec{u} = \langle -2,-2,-2 \rangle / |\vec{PQ}| \rightarrow \vec{u} = \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

$$\underbrace{f_x = 3x^2 + yz, \quad f_y = 3y^2 + xz, \quad f_z = 3z^2 + xy}_{\nabla f(1,1,1) = \langle 4,4,4 \rangle}$$

$D_{\vec{u}} = \vec{u} \cdot \nabla f = \boxed{-4\sqrt{3}}$

7. $g(x,y) = 3x^2 - 3y^2 \quad x = e^u \cos v \;,\; y = e^u \sin v$

$g_x = 6x \quad g_y = -6y \quad x_u = e^u \cos v \quad y_u = e^u \sin v$

$\frac{\partial g}{\partial u} = 6x(e^u \cos v) - 6y(e^u \sin v)$

$\frac{\partial g}{\partial u} = (6e^u \cos v)(e^u \cos v) - (6e^u \sin v)(e^u \sin v)$

$= 6e^{2u} \cos^2 v - 6e^{2u} \sin^2 v$

$= 6\cos^2(1) - 6\sin^2(1) = \boxed{6\cos 2}$

8. $\text{div } F = 3 - 2 + 5 = 6$

$6 \iiint dV \rightarrow 6 \int_{\frac{3\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$

or $6 \times dV \rightarrow \frac{4}{3}\pi r^3 \rightarrow \frac{4}{3}(8)\pi$

only $\frac{1}{8}$ of sphere $\rightarrow \frac{4}{3}\pi$

$6 \times \frac{4}{3}\pi \rightarrow \boxed{8\pi}$

9. $F(x,y,z) = \langle 3z, 2x, y+z \rangle \quad g = 2x + 3y$

$g_x = 2 \quad g_y = 3$

$\iint_D (-3z(2) - 2x(3) + (y+z)\,dA$

$\iint_D -6(2x+3y) - 6x + (2x + 4y)\,dA \rightarrow -12x - 18y - 6x + 2x + 4y$

$\iint_D -16x - 14y \, dA$

$\int_0^1 \int_0^1 -16x - 14y \, dx dy \rightarrow \int_0 -8x^2 - 14xy \Big|_0^1 dy$

$\int_0^1 -8 - 14y \, dy \rightarrow -8y - 7y^2 \Big|_0^1 \rightarrow \boxed{-15}$

10. $f(x,y) = 4x - y^2 - \ln(2x+y)$

$f_x = 4 - \frac{2}{2x+y}$ $f_{xx} = -\frac{4}{(2x+y)^2}$ $f_{xy} = -\frac{4}{(2x+y)^3}$

$f_y = -2y - \frac{1}{2x+y}$ $f_{yy} = -\frac{1}{(2x+y)^2}$

$\left.\begin{array}{l} f_x = 4 - \frac{2}{2x+y} = 0 \\ f_y = -2y - \frac{1}{2x+y} = 0 \end{array}\right\}$ $x = \frac{3}{4}$ $y = -1$

$D = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2$

$D = (-16)(-4) - 1024 = D < 0$

$(\frac{3}{4}, -1)$ is a saddle point

11. $f(x,y,z) = \sqrt{2x^2+3y^2+z^2} = 3$

$f_x = \frac{2x}{(2x^2+3y^2+z^2)}$ $\rightarrow$ $f_x(1,1,2) = \frac{2}{3}$

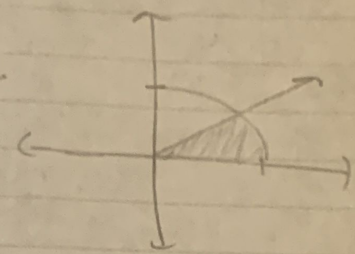
$f_y = 3y/(2x^2+3y^2+z^2)$ $\rightarrow$ $f_y(1,1,2) = 1$

$f_z = z/(2x^2+3y^2+z^2)$ $\rightarrow$ $f_z(1,1,2) = \frac{2}{3}$

$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$

$L(1.001, .999, 2.001) = \boxed{\frac{9001}{3000}}$

12. $\underbrace{\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx}$

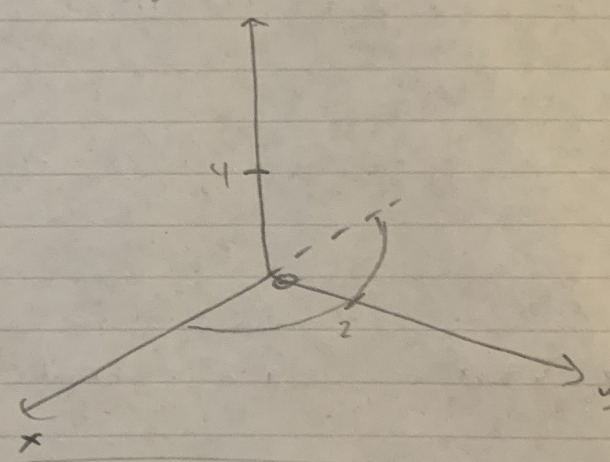


$\int_0^{\frac{\pi}{4}} \int_0^{1} r^2 \cos\theta \, dr\, d\theta$

$\int_0^{\frac{\pi}{4}} \frac{r^3}{3} \cos\theta \Big|_0^1 \, d\theta$

$\int_0^{\frac{\pi}{4}} \frac{1}{3} \cos\theta \, d\theta \rightarrow \frac{1}{3} \sin\theta \Big|_0^{\pi/4} = \boxed{\frac{\sqrt{2}}{6}}$

13. $\int_0^{2} \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 y z \, dx\,dy\,dz$



$\boxed{\int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2} (\rho\sin\phi\cos\theta)^2 (\rho\sin\phi\sin\theta)(\rho\cos\phi)\, \rho^2\sin\phi \, d\rho\, d\phi\, d\theta}$

14. $r(t) = \langle 5, 3\sin t, 3\cos t \rangle$ $\quad K = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3}$ $\quad t = \frac{\pi}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$ $\quad r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$r'(t) \times r''(t) = \langle -9, 0, 0 \rangle$

$|r'(t) \times r''(t)| = 9$ $\quad |r'(t)| = 3$

$\frac{9}{(3)^3} = \frac{9}{27} = \boxed{\frac{1}{3}}$

15. $r(u,v) = \langle u^2, uv, v^2 \rangle \quad 0 \le u < v < 1$

$$\left.\begin{array}{l} r_u = \langle 2u, v, 0 \rangle \\ r_v = \langle 0, u, 2v \rangle \end{array}\right\} r_u \times r_v = \langle 2v^2, -4uv, 2u^2 \rangle$$

$$|r_u \times r_v| = |\langle 2v^2, -4uv, 2u^2 \rangle|$$
$$= 2\sqrt{v^4 + u^4 + 4v^2u^2}$$

$$\iint 2\sqrt{v^4 + u^4 + 4v^2u^2}\, dv\, du$$

$$\int_0^1 \int_u^1 2\sqrt{v^4 + u^4 + 4v^2u^2}\, dv\, du$$

4. $a \times b = i + j - k$, $b \times c = i - j + k$, $a \times c = 2i + j + 2k$

$$(a+b+c) \times (2a - b + 3c)$$

$$u \times (v + w) = u \times v + u \times w$$

Use the identities

16. $f(x,y,z) = xy^2z^3 \quad g(x,y,z) = x + y^2 + z^3$

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ at $(1,1,1) = \langle 1,2,3 \rangle$
$\nabla g = \langle 1, 2y, 3z^2 \rangle$ at $(1,1,1) = \langle 1,2,3 \rangle$

$\langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle = \boxed{14}$

17. $\lim_{(x,y,z,w) \to (0,0,0,0)} \frac{(x+y)^2 - (z+w)^2}{x+y-z-w} = \frac{0}{0}$

On x-axis, $\to \frac{x^2}{x} = x$
On y-axis, $\to \frac{y^2}{y} = y$ $= \boxed{\text{lim DNE}}$

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II / I and II
MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020
Email the completed test, renamed as finalFirstLast.pdf to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).
WRITE YOUR FINAL ANSWERS BELOW
1. -18
2. Int(Int(f(x,y)) x=y^2..1) 1/2..1)
3. z = (-1/2)x - (5/6)y + 7pi/18
4.
5. Angle A = pi/3, Angle B = pi/3, Angle 3 = pi/3
6. -4sqrt3
7. 6cos2
8. 8pi
9. -15
10. (3/4, -1) is a saddle point
11. 9001/3000
12. (Sqrt2)/6
13. Int(_Int(Int((psinphicostheta)^2(psinphisintheta)(pcosphi)(p^2sin phi)) p=0..2). Phi=0..pi/2) theta=pi/2..pi)
14. 1/3
15. Int(Int(2sqrt(v^4+u^4+4v^2u^2)) v=u..1) u=0..1
16. 14
17. DNE

Sign the following declaration:
I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but not

other resources on the internet.
I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.
I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.
Signed: Daniel Lim