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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. $a^4 \ln 2 - 3\sin a$
2. $D = \{(x,y) \mid 1/2 \le y \le 1, 4y^2 \le x \le 1\}$
3. $z = 0 + -3\sqrt{3}(x - \frac{\pi}{3}) + -5\sqrt{3}(y - \frac{\pi}{3}) + -6\sqrt{3}(z - \frac{\pi}{3})$
4. $-i - 7j + 5k$
5. $A = \frac{\pi}{3}$, $B = \pi/3$ $C = \pi/3$ all equal distance
6. $-4\sqrt{3}$
7. $6\cos^2(1) - 6\sin^2(1)$
8. $\frac{342\pi}{2}$
9. -15
10. $x = 3/4$ $y = -1$
11. $f_x = 2/3$ $f_y = 1$ $f_z = 2/3$, $L(x,y,z) = 3 + 2/3(x-1) + 1(y-1) + 2/3(z-2)$
12. $P(r_0 \cdot S, \theta\theta)$
13.
14. $1/3$
15. $\|N\| = \sqrt{4v^4 + 16u^2v^2 + 4v^2}$
16. 14
17. 0

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Jessuma Betts

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (\cos(e^{\sin x}) + 5y)\, dx + (\sin(e^{\cos y}) + 11x)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans.

$-\iint_R (11-5)\,dA$

$\int_C \vec{F} \cdot dr$ $\quad f(x,y) = xy\,i' + 11y\,j$

$\int_0^1$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, \cancel{dx\,dy} \, .$$

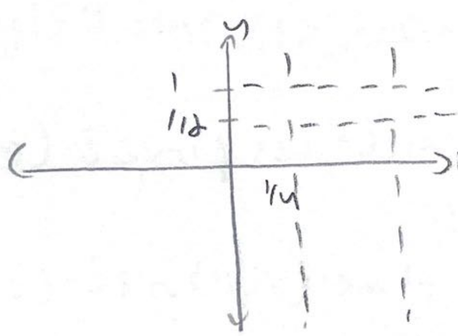
$dy\, dx$

---

ans.

$D = \{ (x,y) \mid 0 \le y \le \sqrt{x} \, , \ \frac{1}{4} \le x \le 1 \}$

$D = \{ (x,y) \mid 1/2 \le y \le 1, \ y^2 \le x \le 1 \}$



3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

ans. $z =$

$2\cos(\frac{\pi}{3}) + 4\cos(\pi/3) + 8\cos(\frac{\pi}{3}) = 7$

$1 + 2 + 4 = 7$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x = -2\sin(x+y) - 4\sin(x+z) - 2\sin(\frac{\pi}{3}) - 4\sin(\pi/3)$

$f_y = -2\sin(x+y)\ 8\sin(y+z) - 2\sin(\pi/3) - 8\sin(\pi/3)$

$f_z = -4\sin(x+z) - 8\sin(y+z) - 4\sin(\pi/3) - 8\sin(\pi/3)$

$\boxed{z = 0 + -3\sqrt{3}(x - \frac{\pi}{3}) + -5\sqrt{3}(y - \frac{\pi}{3}) + -6\sqrt{3}(z - \pi/3)}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans.

$i \quad j \quad k$

$a \quad b \quad c$

$2a \quad -b \quad 3c$

$i(3bc - bc) - j(3ac - 2ac) + k(-ab - 2ab)$

$= 3(i - j + k) + i - j + k$

$3i - 3j + 3k + i - j + k$

$4i - 4j + 4k$

$= 3(2i + j + 2k) - 2(2i + j + 2k)$

$6i + 3j + 6k - 4i + 2j + 4k$

$-2i + j + 2k$

$= -1(i + j - k) - 2(i + j - k)$

$= -i - j + k - (2i + 2j - 2k)$

$= -3i - 2j + 3k$

$(2i - 5j + 2k) + -3i + -2j + 3k = -i - 7j + 5k$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: radians ;

The angle at B is: radians ;

The angle at C is: radians ;

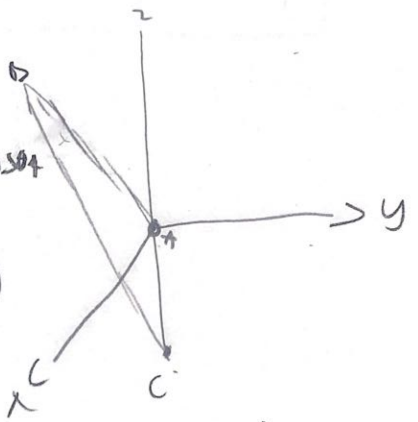
$R_{AB} = \langle 1, 0, 1 \rangle$

$R_{AC} = \langle 1, 1, 0 \rangle$

$R_{BC} = \langle 0, 1, -1 \rangle$

$\vec{r}_{AB} \cdot \vec{r}_{AC} = ||R_{AB}|| \, ||r_{AC}|| \cos\theta_A$

$\theta_A = \cos^{-1}\left(\frac{R_{AB} \cdot r_{AC}}{||r_{AB}|| \, || \, ||R_{AC}||}\right)$



$R_{AB} = \langle 1, 0, 1 \rangle$

$R_{AC} = \langle 1, 1, 0 \rangle$

$R_{BC} = \langle 0, 1, -1 \rangle$

$\theta_A = \pi/3$

$\theta_B = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta_C = \cos^{-1}\left(\frac{1}{2}\right)$

$\Big\} = \pi/3$

all equal distance

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

---

ans. $-4\sqrt{3}$

number

$|\vec{v}| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

$\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$

$D(x^3 + xyz + y^3 + z^3) \text{ @ } (1,1,1) = (4,4,4)$

$\left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right) = \boxed{-4\sqrt{3} \approx -6.9}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans.

$$\frac{dg}{du} = \frac{dg}{dx} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{dg}{dx} = 6x \qquad \frac{dg}{dy} = -6y$$

$$\frac{\partial x}{\partial u} = e^v \cos v \qquad \frac{dy}{du} = e^v \sin v$$

$$\frac{dg}{du} = 6e^v \cos v \cdot e^v \cos(v) - 6e^v \sin(v) \cdot e^v \sin(v)$$

$$= 6e^{2v} \cos^2(v) - 6e^{2v} \sin^2(v)$$

$$\left.\frac{\partial g}{\partial u}\right|_{(0,1)} = 6\cos^2(1) - 6\sin^2(1)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans.

$D = \{ \rho, \theta, \phi \mid 0 \leq \rho \leq 2 \,,\, 0 \leq \phi \frac{\pi}{2},$

$x = \rho \sin\phi \cos\theta \quad y = \rho \sin\theta \sin\phi$

$z = \rho \cos\phi \quad dv = \rho^2 \sin\phi \, d\rho \, d\phi$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans.

$G(u,v) = \langle u, v, 2u + 3v \rangle \to F(G(u,v)) = \langle 3(2u+3v), 2u, v + 2u+3v \rangle$

$= \langle 6u + 9v, 2u + 2u \rangle$

$G_u = \langle 1, 0, 2 \rangle$

$G_v = \langle 0, 1, 3 \rangle$

$N = G_u \times G_v = \langle -2, -3, 1 \rangle$

$\int_0^1 \int_0^1 F(G(u,v)) \cdot N \, du\,dv$

$= \int_0^1 \int_0^1 \langle 6u + 9v, 2u, 2u + 4v \rangle \langle -2, -3, 1 \rangle \, du\,dv$

dot product $= -15$

take duble integral and dot product

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

ans. no critical points

$f(x,y) = 4x - y^2 - \ln(2x+y)$

$f_x = 4 - \frac{2}{2x+y}$

$f_{xx} = \frac{4}{(2x+y)^2}$

$f_y = -2y - \frac{1}{2x+y}$

$f_{yy} = \frac{1}{(2x+y)^2} - 2$

$f_x = 0 = 4 - \frac{2}{2x+y} = 0$

$4x + 2y = 1$

$y = \frac{1-4x}{2}$

$f_y = 0 = -2y - \frac{1}{2x+y} = 0$

$2y + \frac{1}{2x+y} = 0$

$2y = \frac{-1}{2x+y}$

$2y(2x+y) = -1$

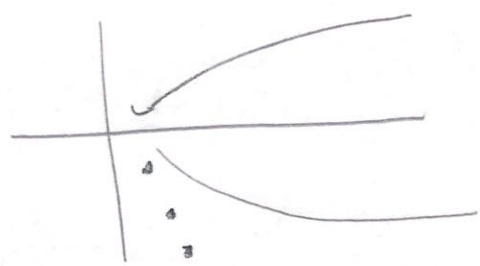
$2\left(\frac{1-4x}{2}\right)\left(2x + \frac{1-4x}{2}\right) = -1$

$(1-4x)\left(2x + \frac{1-4x}{2}\right) = -1$

$0 \neq -1$

$x = -3/4$

$y = 1$



$f'(x+y) = \frac{4(2x+y)-2}{2x+y}$

$\frac{4(2(3/4)+1)-2}{2(-3/4)+1}$

$\frac{4(6/4+1)-2}{-6/4+1}$

$-\frac{24}{4} + 4 - 2$

$\frac{6+4-2}{6/4+1}$

$\frac{6+4-2y}{6/4+1}$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

ans.

$P = (1, 1, 2)$

$F_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x \quad \frac{2}{\sqrt{3+2+2^2}} = \frac{2}{\sqrt{2+3+4}} = \frac{2}{\sqrt{9}} = 2/3$ ✓

$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y = \frac{3}{\sqrt{2+3+2^2}} = 1$ ✓

$f_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 2z \to \frac{2}{\sqrt{2+3+2^2}} = 2/3$ ✓

$L(x, y, z) = 3 + 2/3(x-1) + 1(y-1) + 2/3(z-2)$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain!

---

ans.

$D_1 = \{ (x,y) \,|\, 0 \le y \le x \quad 0 \le x \le \frac{\sqrt{2}}{2} \}$

$y = x \quad r\sin\theta = r\cos\theta$

$\tan\theta = 1 \quad \theta = \frac{\pi}{4} \quad r\cos\theta = \frac{\sqrt{2}}{2} \to \theta \frac{\pi}{4}$

$r\cos\theta = 0 \quad \theta = \frac{\pi}{2} \quad \tan\theta = 1 \; \theta = \frac{\pi}{4}$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to spherical coordinates. Do not evaluate.

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ans.

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14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $1/3$, number

$$\kappa(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$r'(t) = \langle 0, 3\cos(t), -3\sin(t)\rangle$

$r''(t) = \langle 0, -3\sin(t), -3\cos(t)\rangle$

$r'(\pi/3) = \langle 0, 3/2, \frac{-3\sqrt{3}}{2}\rangle$

$r''(\pi/3) = \langle 0, \frac{-3\sqrt{3}}{2}, -3/2\rangle$

$$\frac{\|\langle -9, 0, 0\rangle\|}{\|\langle 0, 3/2, \frac{-3\sqrt{3}}{2}\rangle\|^3} = \frac{9}{3^3} = 1/3$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u, v) = \langle u^2 , uv , v^2 \rangle \quad , \quad 0 < u < v < 1 \ .$$

---

ans.

$$\int_0^1 \int_0^1 \sqrt{4v^4 + 16u^2v^2 + 4u^4}\, du\, dv$$

$r_u = \langle 2u, v, 0 \rangle$

$r_v = \langle 0, u, 2v \rangle$

$N = r_u \times r_v = \begin{bmatrix} 2u \\ v \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ u \\ 2v \end{bmatrix} = \begin{bmatrix} 2v^2 - 0 \\ -4uv \\ 2u^2 - 0 \end{bmatrix} =$

$\langle 2v^2, -4uv, 2u^2 \rangle$

$$\boxed{\|N\| = \sqrt{4v^4 + 16u^2v^2 + 4u^8}}$$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \quad .$$

at the point $(1, 1, 1)$.

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ans. 14 , number

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle \to \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle \to \langle 1, 2, 3 \rangle$

$\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1 + 2^2 + 3^2 = 14$

Answer = 14

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. number

Answer = 0

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2}{x+y-z-w} = \lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(z+w)^2}{x+y-z-w}$$

used maple