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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 18

2. Int(Int(f(x,y), x=1..1/4), y=0..1/2)+Int(Int(f(x,y), x=y^2..1), y=1/2..1)

3. z = -3*sqrt(3)*x-5*sqrt(3)*y+Pi*(4*sqrt(3)/3+1/6)

4. -i-4j-9k

5. ans. The angle at A is:45 radians: Pi/4

The angle at B is:90 Radians: Pi/2 ;

The angle at C is:45 Radians: Pi/4 ;

6. -4*sqrt(3)

7.0

8. 8Pi

9. -15

10. (3/4,-1) a saddle point

11. 3.0003

12. – (sqrt(2)-2)/6

13. Int(Int(Int((rsin(p)cos(t))^2*(rsin(p)sin(t))*(rcos(p))*r^2*sin(p), r=0..2), p=3Pi/4..Pi), t=3Pi/2..2Pi)

14. 3*sqrt(2)/2

15. Int(Int(iu^2+juv+kv^2, u=0..v), v=0..1)

16. 14

17.0

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Yongshan Li

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \,.$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos(e^{\sin x}) + 5y)\, dx + (sin(e^{\cos y}) + 11x)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans. 18

1.

$P = (\cos(e^{\sin(x)}) + 5y) \quad Q = (\sin(e^{\cos(y)}) + 11x)$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(\sin(e^{\cos(y)}) + 11x) - \frac{\partial}{\partial y}(\cos(e^{\sin(x)}) + 5y)$$

$= 6$

$\iint_D 6 \, dA$

$= 6 \times$ (Area of the bound)

$= 6 \times 2 \times 1 + 6 \times 2 \times 1 \times \frac{1}{2}$

$= 18$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx \quad .$$

---

ans. Int(Int(f(x,y), x=1..1/4), y=0..1/2)+Int(Int(f(x,y), x=y^2..1), y=1/2..1)

2.

$D=\{(x,y)\mid \frac{1}{4}\le x\le 1,\ 0\le y\le\sqrt{x}\}$

$D=\{(x,y)\mid (0\le y\le\frac{1}{2},\ \frac{1}{4}\le x\le 1)\cup(\frac{1}{2}\le y\le 1,\ y^2\le x\le 1)\}$

$\int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy+\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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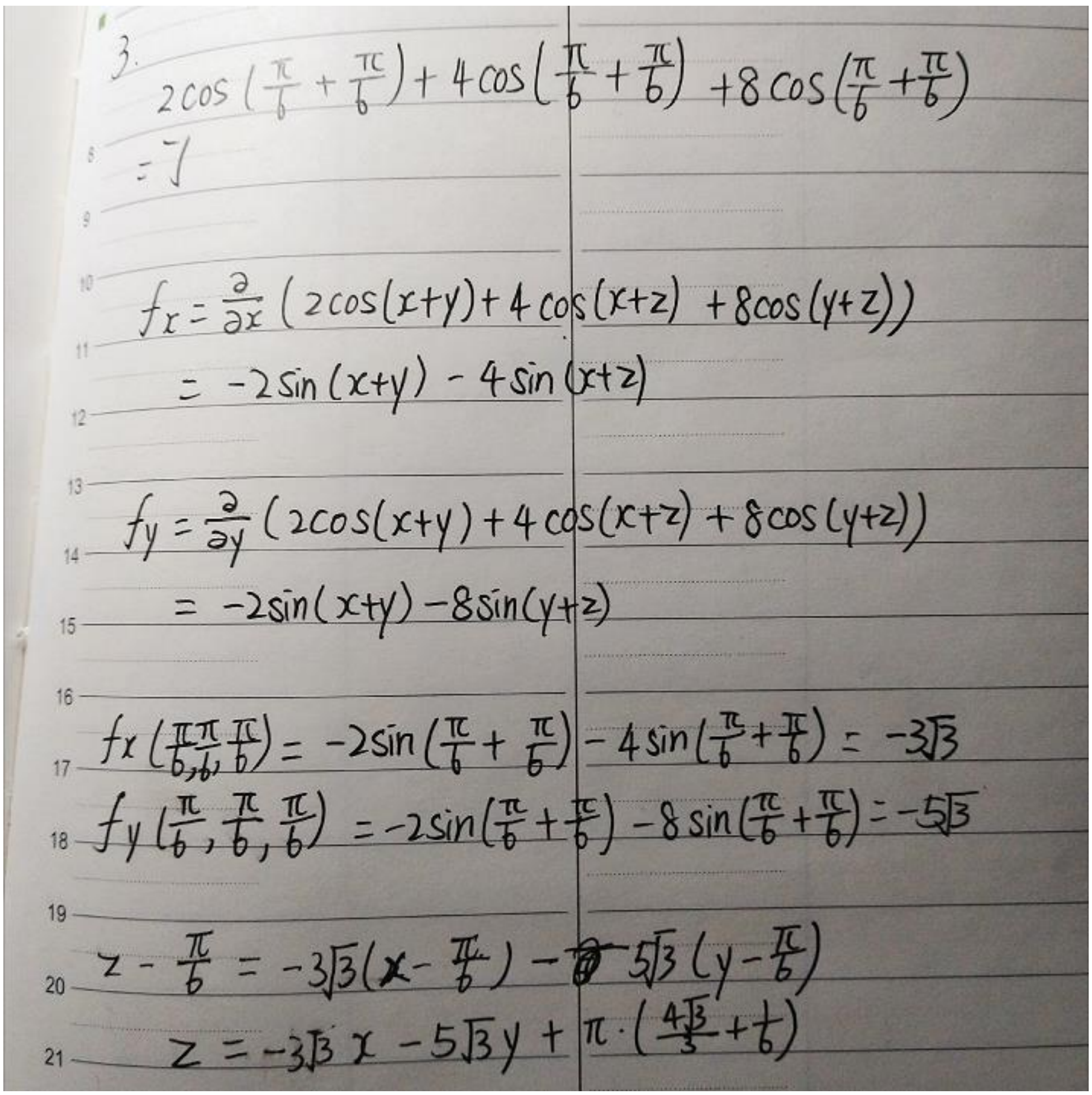
ans. z = -3*sqrt(3)*x-5*sqrt(3)*y+Pi*(4*sqrt(3)/3+1/6)



3.

$2\cos(\frac{\pi}{6}+\frac{\pi}{6}) + 4\cos(\frac{\pi}{6}+\frac{\pi}{6}) + 8\cos(\frac{\pi}{6}+\frac{\pi}{6})$

$= 7$

$f_x = \frac{\partial}{\partial x}(2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z))$

$= -2\sin(x+y) - 4\sin(x+z)$

$f_y = \frac{\partial}{\partial y}(2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z))$

$= -2\sin(x+y) - 8\sin(y+z)$

$f_x(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}) = -2\sin(\frac{\pi}{6}+\frac{\pi}{6}) - 4\sin(\frac{\pi}{6}+\frac{\pi}{6}) = -3\sqrt{3}$

$f_y(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6}) = -2\sin(\frac{\pi}{6}+\frac{\pi}{6}) - 8\sin(\frac{\pi}{6}+\frac{\pi}{6}) = -5\sqrt{3}$

$z - \frac{\pi}{6} = -3\sqrt{3}(x-\frac{\pi}{6}) - 5\sqrt{3}(y-\frac{\pi}{6})$

$z = -3\sqrt{3}x - 5\sqrt{3}y + \pi\cdot(\frac{4\sqrt{3}}{3}+\frac{1}{6})$

4. (16 points) Let **a**, **b**, **c** be three vectors such that

$$\mathbf{a}\times\mathbf{b}=\mathbf{i}+\mathbf{j}-\mathbf{k} \quad , \quad \mathbf{b}\times\mathbf{c}=\mathbf{i}-\mathbf{j}+\mathbf{k} \quad , \quad \mathbf{a}\times\mathbf{c}=2\mathbf{i}+\mathbf{j}+2\mathbf{k} \quad .$$

What is

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\times(2\mathbf{a}-\mathbf{b}+3\mathbf{c}) \qquad ?$$

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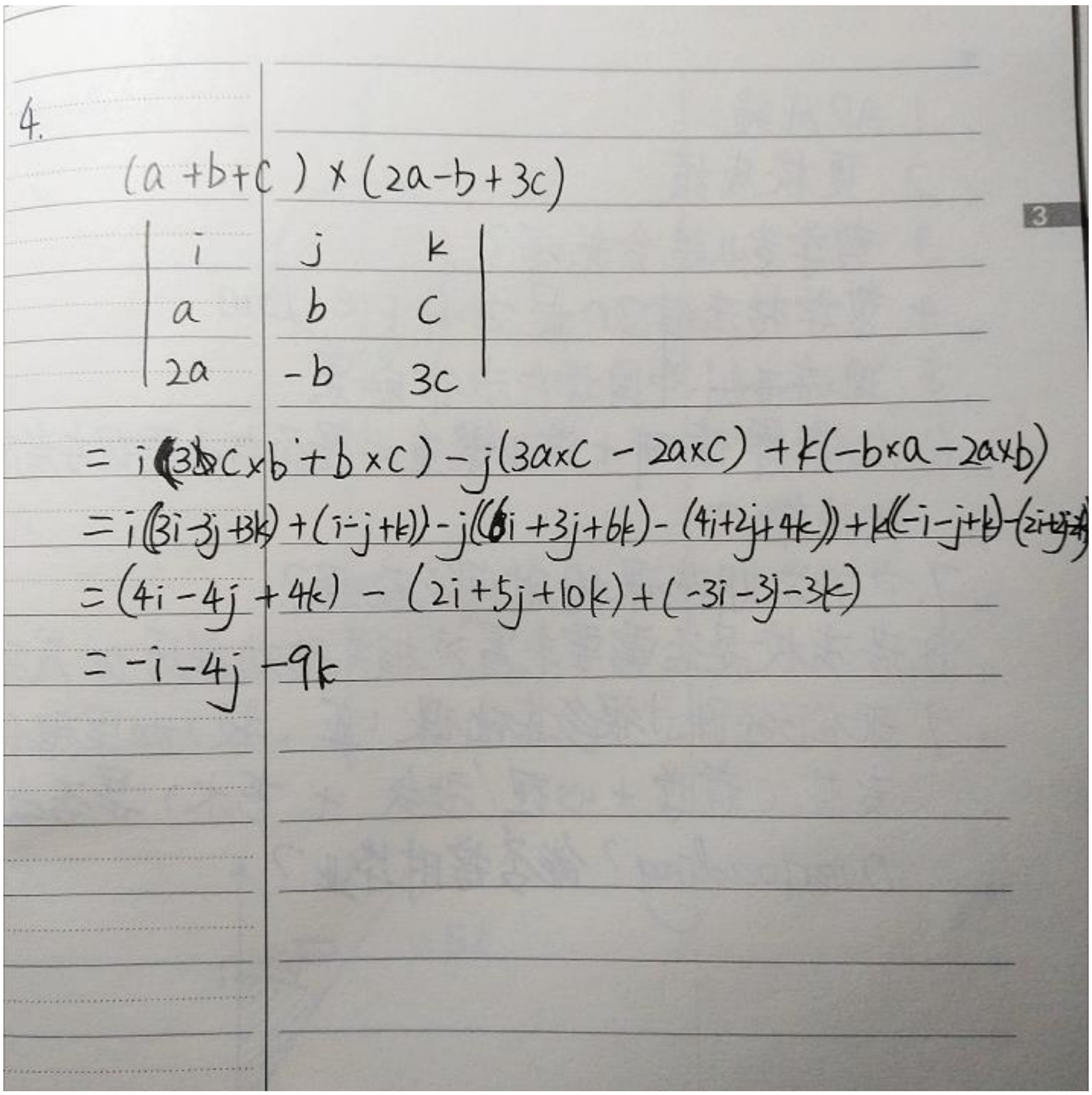
ans. -i-4j-9k



5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is:45 radians: Pi/4 ;

The angle at B is:90 Radians: Pi/2 ;

The angle at C is:45 Radians: Pi/4 ;

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6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

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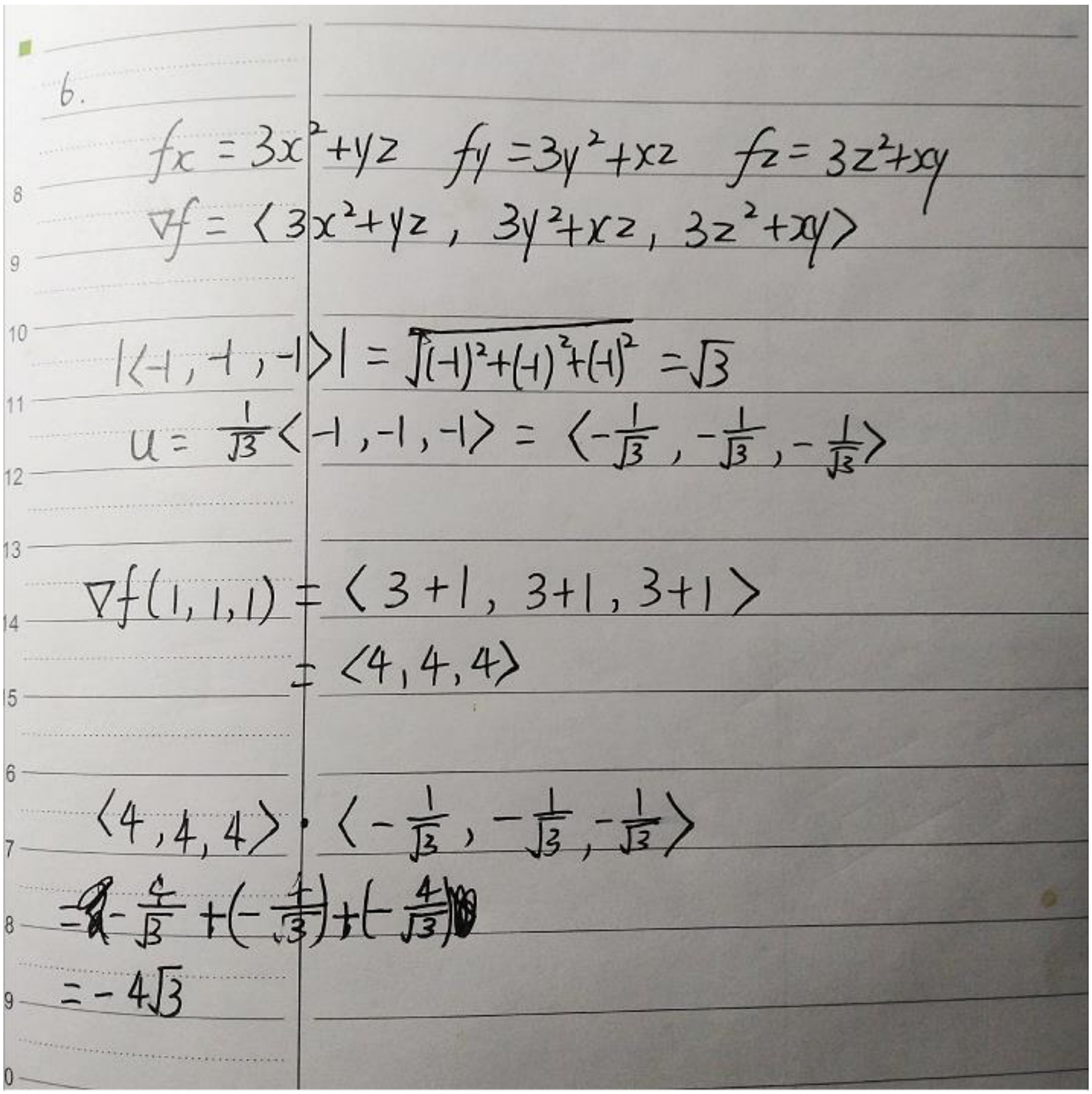
ans. -4*sqrt(3)



6.

$f_x = 3x^2 + yz \quad f_y = 3y^2 + xz \quad f_z = 3z^2 + xy$

$\nabla f = \langle 3x^2 + yz,\ 3y^2 + xz,\ 3z^2 + xy \rangle$

$|\langle -1, -1, -1 \rangle| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$

$u = \frac{1}{\sqrt{3}} \langle -1, -1, -1 \rangle = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$\nabla f(1,1,1) = \langle 3+1,\ 3+1,\ 3+1 \rangle$

$= \langle 4, 4, 4 \rangle$

$\langle 4, 4, 4 \rangle \cdot \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$= -\frac{4}{\sqrt{3}} + (-\frac{4}{\sqrt{3}}) + (-\frac{4}{\sqrt{3}})$

$= -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

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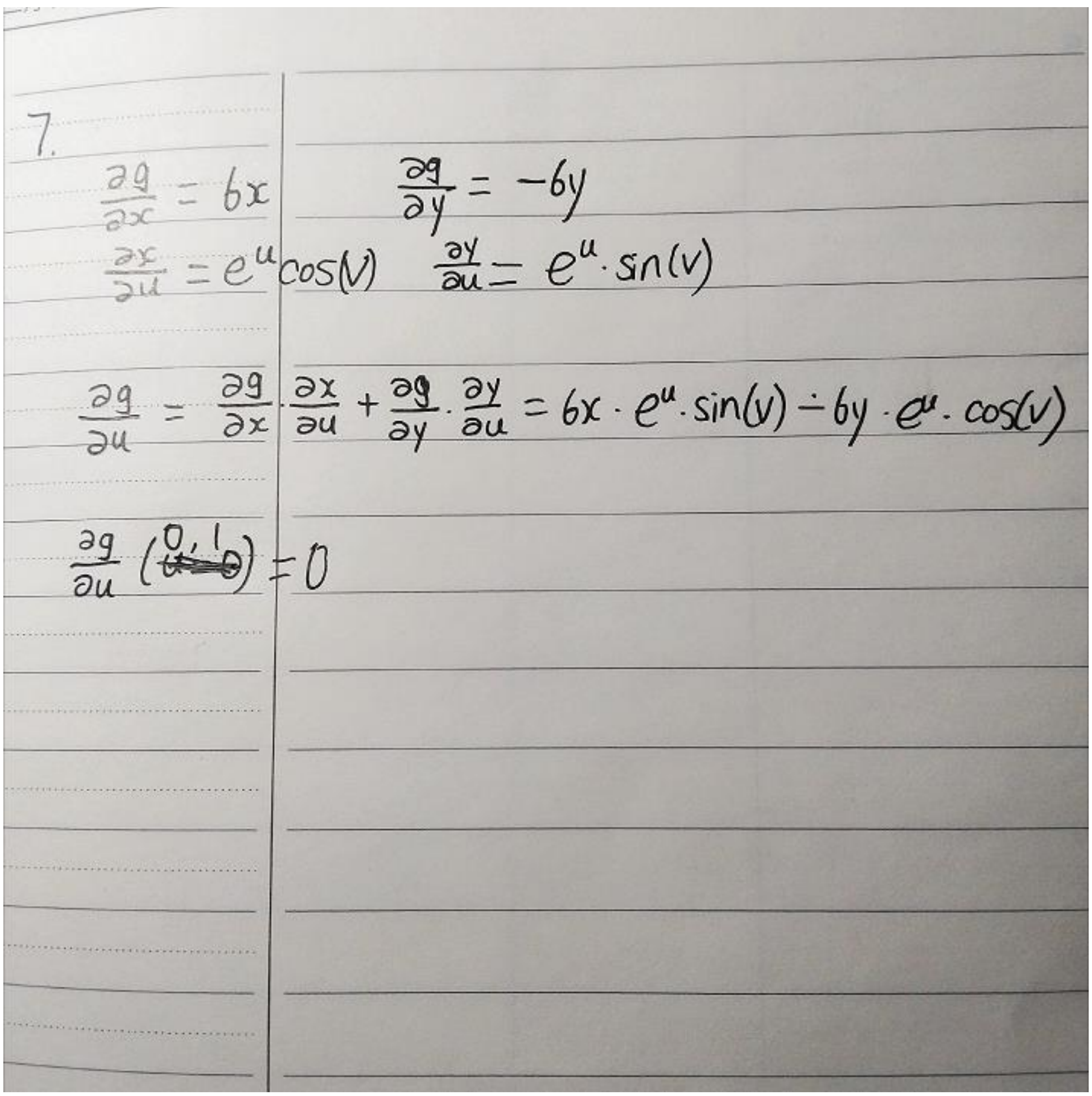
ans. 0



7.

$\frac{\partial g}{\partial x} = 6x \qquad \frac{\partial g}{\partial y} = -6y$

$\frac{\partial x}{\partial u} = e^u \cos(v) \qquad \frac{\partial y}{\partial u} = e^u \cdot \sin(v)$

$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u} = 6x \cdot e^u \cdot \sin(v) - 6y \cdot e^u \cdot \cos(v)$

$\frac{\partial g}{\partial u}(0, 1) = 0$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\iint_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = (3x + cos(y^3 + yz),\ -2y + e^{x+z^2},\ 5z + \sin(xy^3 + e^x)) \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8Pi

8.

$\operatorname{div} F = \frac{\partial}{\partial x}(3x + \cos(y^3 + yz)) + \frac{\partial}{\partial y}(-2y + e^{x+z^2}) + \frac{\partial}{\partial z}(5z + \sin(xy^3 + e^x))$

$= 3 - 2 + 5 = 6$

$D = \{x^2 + y^2 + z^2 < 4,\ x > 0,\ y < 0,\ z > 0\}$

$\iint_S F \cdot dS = 6 \times$ (Volume of the bound)

$= 6 \times \frac{1}{8} \times \frac{4}{3} \times \pi \times 2^3$

$= 8\pi$

9. (12 points) Compute the vector-field surface integral $\iint_S \mathbf{F} . d\mathbf{S}$ if

$$\mathbf{F} = (3z, 2x, y+z) \quad ,$$

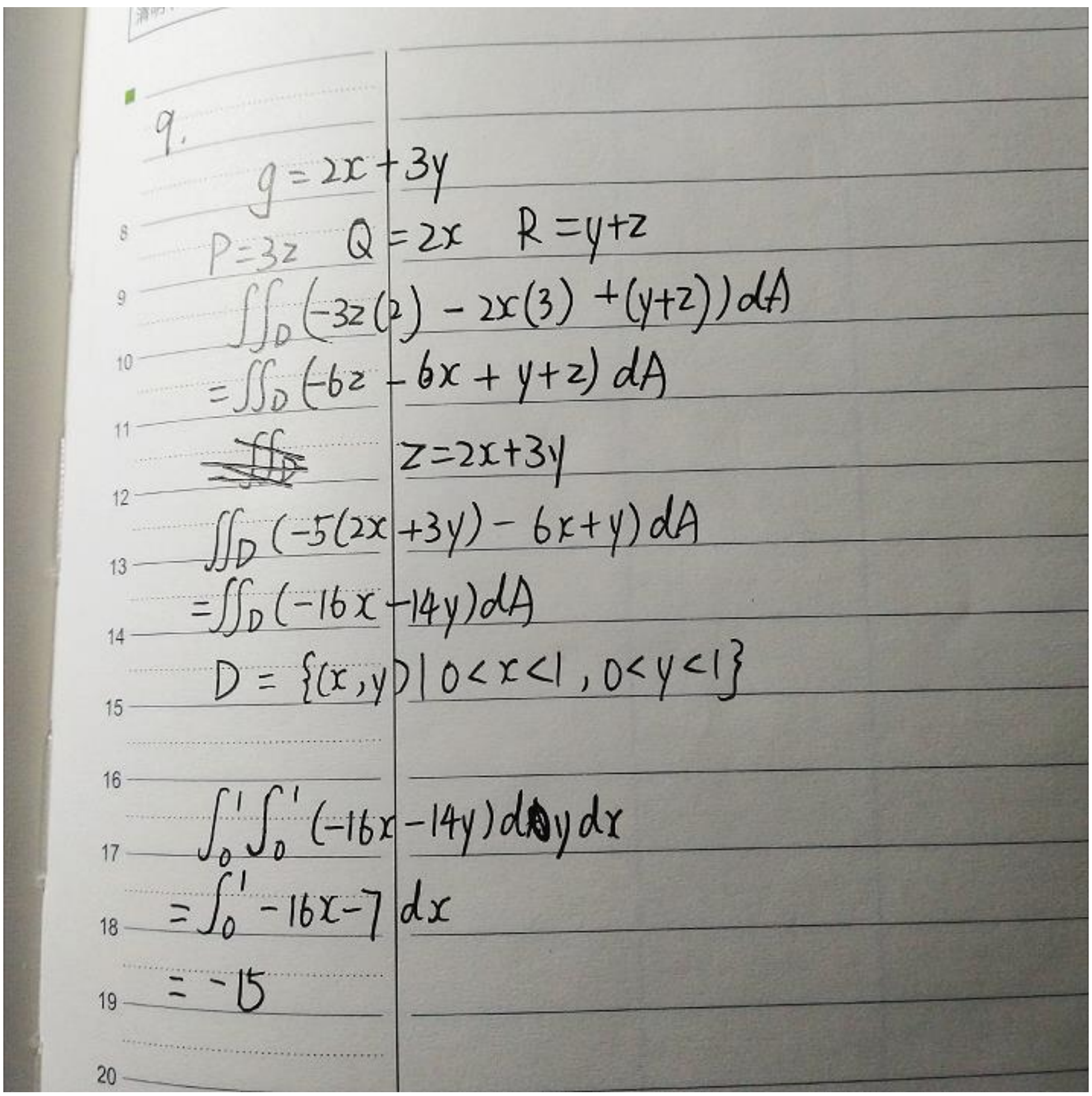
and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15



9.

$g = 2x + 3y$

$P = 3z \quad Q = 2x \quad R = y+z$

$\iint_D (-3z(2) - 2x(3) + (y+z))\, dA$

$= \iint_D (-6z - 6x + y + z)\, dA$

$z = 2x + 3y$

$\iint_D (-5(2x+3y) - 6x + y)\, dA$

$= \iint_D (-16x - 14y)\, dA$

$D = \{(x,y) \mid 0 < x < 1, 0 < y < 1\}$

$\int_0^1 \int_0^1 (-16x - 14y)\, dy\, dx$

$= \int_0^1 -16x - 7 \, dx$

$= -15$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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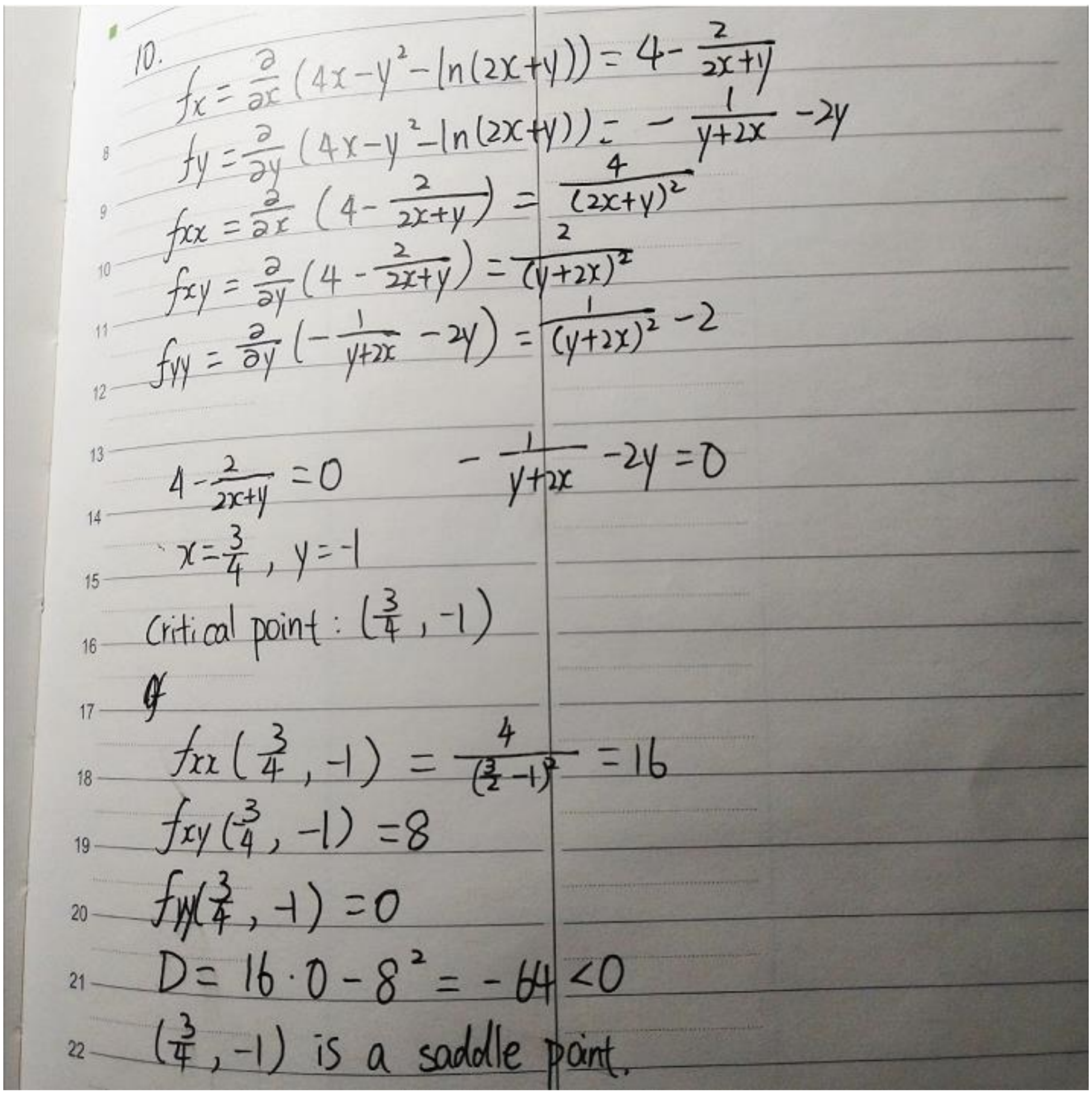
ans. (3/4,-1) a saddle point



10.

$f_x = \frac{\partial}{\partial x}(4x - y^2 - \ln(2x+y)) = 4 - \frac{2}{2x+y}$

$f_y = \frac{\partial}{\partial y}(4x - y^2 - \ln(2x+y)) = -\frac{1}{y+2x} - 2y$

$f_{xx} = \frac{\partial}{\partial x}\left(4 - \frac{2}{2x+y}\right) = \frac{4}{(2x+y)^2}$

$f_{xy} = \frac{\partial}{\partial y}\left(4 - \frac{2}{2x+y}\right) = \frac{2}{(y+2x)^2}$

$f_{yy} = \frac{\partial}{\partial y}\left(-\frac{1}{y+2x} - 2y\right) = \frac{1}{(y+2x)^2} - 2$

$4 - \frac{2}{2x+y} = 0 \qquad -\frac{1}{y+2x} - 2y = 0$

$x = \frac{3}{4}, \ y = -1$

Critical point: $(\frac{3}{4}, -1)$

$f_{xx}(\frac{3}{4}, -1) = \frac{4}{(\frac{3}{2} - 1)^2} = 16$

$f_{xy}(\frac{3}{4}, -1) = 8$

$f_{yy}(\frac{3}{4}, -1) = 0$

$D = 16 \cdot 0 - 8^2 = -64 < 0$

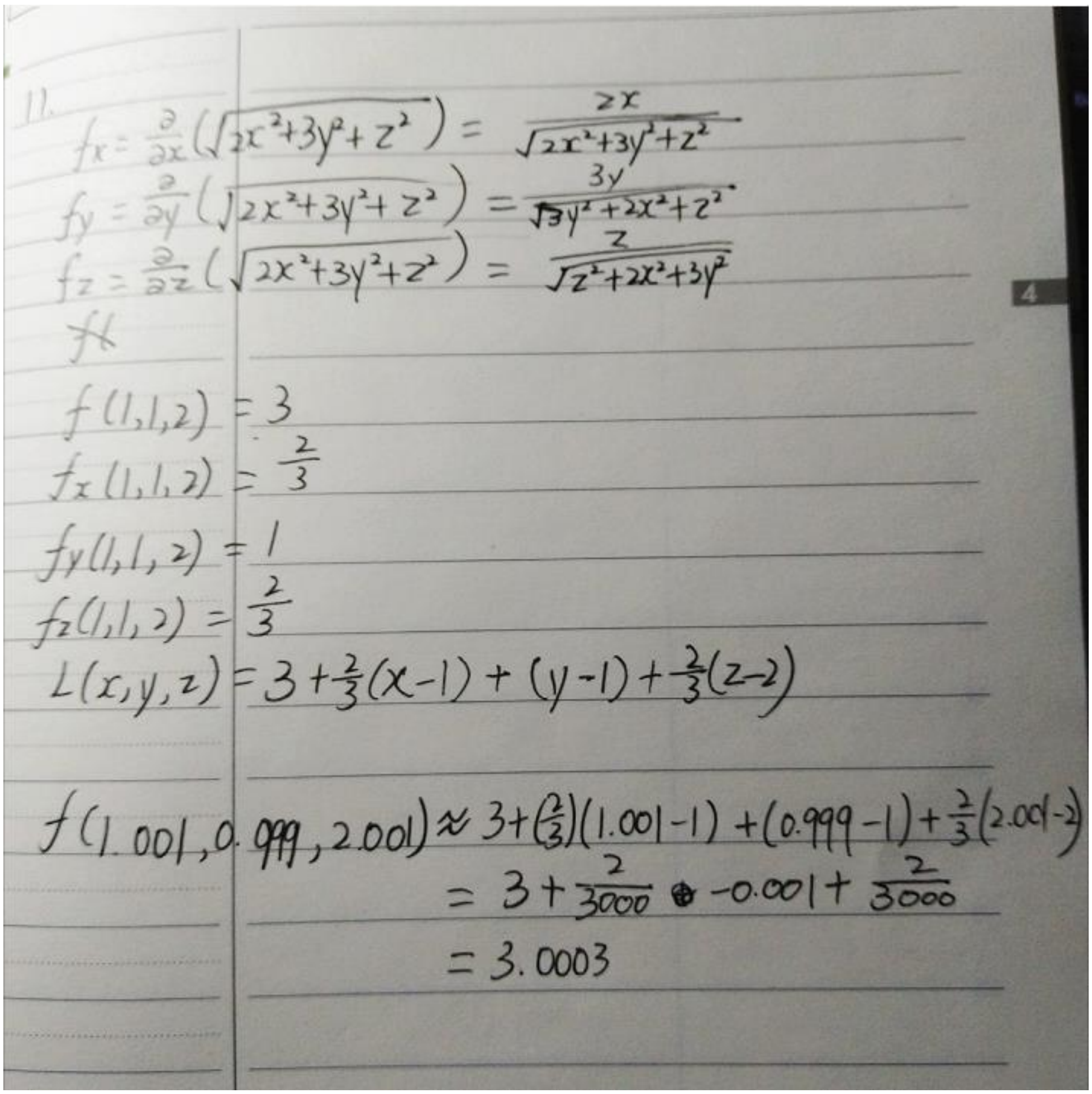
$(\frac{3}{4}, -1)$ is a saddle point.

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.0003



11.

$$f_x = \frac{\partial}{\partial x}\left(\sqrt{2x^2+3y^2+z^2}\right) = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_y = \frac{\partial}{\partial y}\left(\sqrt{2x^2+3y^2+z^2}\right) = \frac{3y}{\sqrt{3y^2+2x^2+z^2}}$$

$$f_z = \frac{\partial}{\partial z}\left(\sqrt{2x^2+3y^2+z^2}\right) = \frac{z}{\sqrt{z^2+2x^2+3y^2}}$$

$$f(1,1,2) = 3$$

$$f_x(1,1,2) = \frac{2}{3}$$

$$f_y(1,1,2) = 1$$

$$f_z(1,1,2) = \frac{2}{3}$$

$$L(x,y,z) = 3 + \frac{2}{3}(x-1) + (y-1) + \frac{2}{3}(z-2)$$

$$f(1.001, 0.999, 2.001) \approx 3 + \left(\frac{2}{3}\right)(1.001-1) + (0.999-1) + \frac{2}{3}(2.001-2)$$

$$= 3 + \frac{2}{3000} - 0.001 + \frac{2}{3000}$$

$$= 3.0003$$

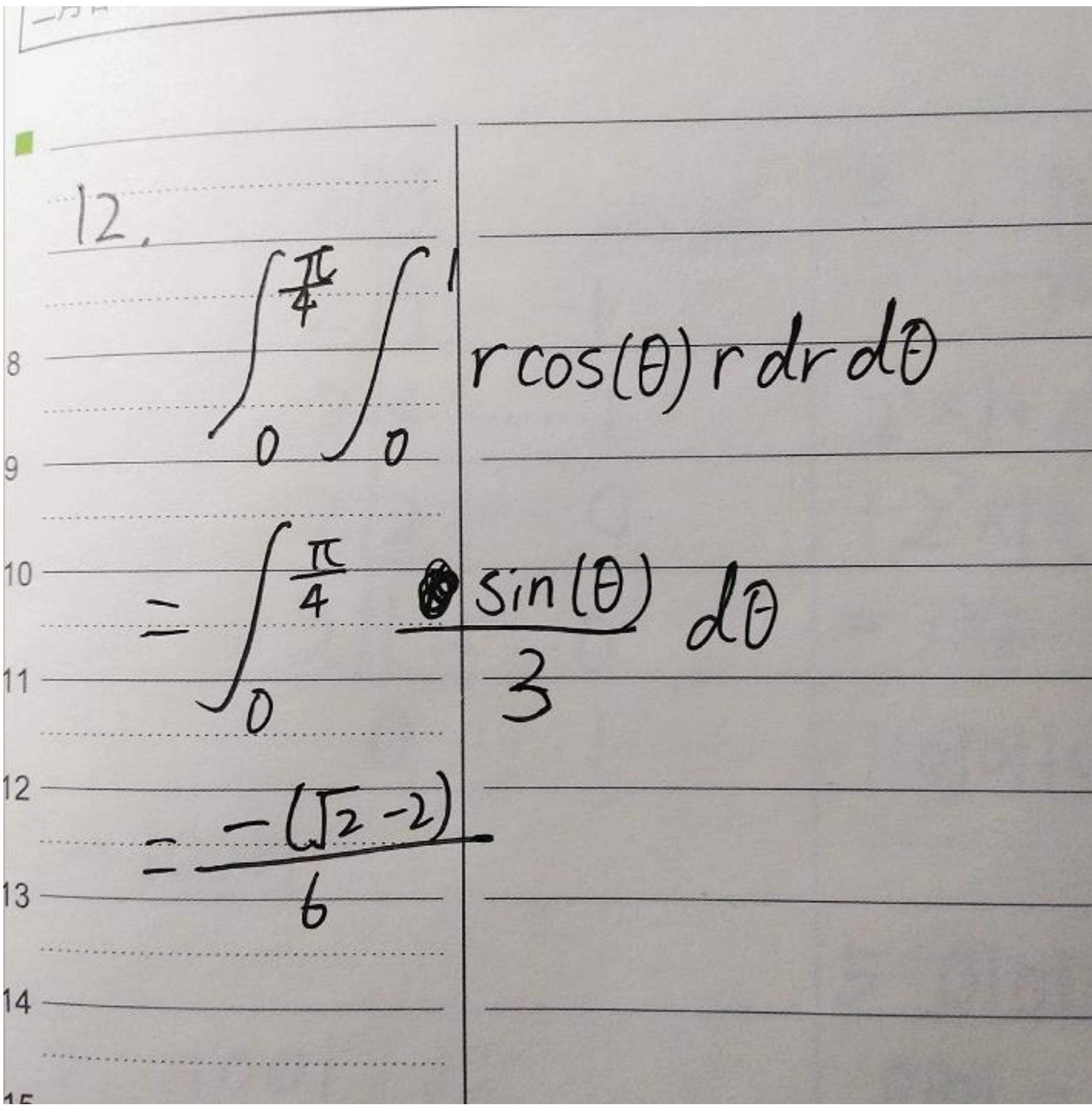
12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\, dy\, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\, dy\, dx \quad .$$

Explain!

---

ans. –(sqrt(2)-2)/6



13. (12 points) Convert the triple iterated integral

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. Int(Int(Int((rsin(p)cos(t))^2*(rsin(p)sin(t))*(rcos(p))*r^2*sin(p), r=0..2), p=3Pi/4..Pi), t=3Pi/2..2Pi)

13.

$$\int_{\frac{3}{2}\pi}^{2\pi} \int_{\frac{3}{4}\pi}^{\pi} \int_0^2 (\rho \sin\varphi \cos\theta)^2 \cdot (\rho \sin\varphi \sin\theta) \cdot (\rho \cos\varphi)\, \rho^2 \sin\varphi \; d\rho\, d\varphi\, d\theta$$

4

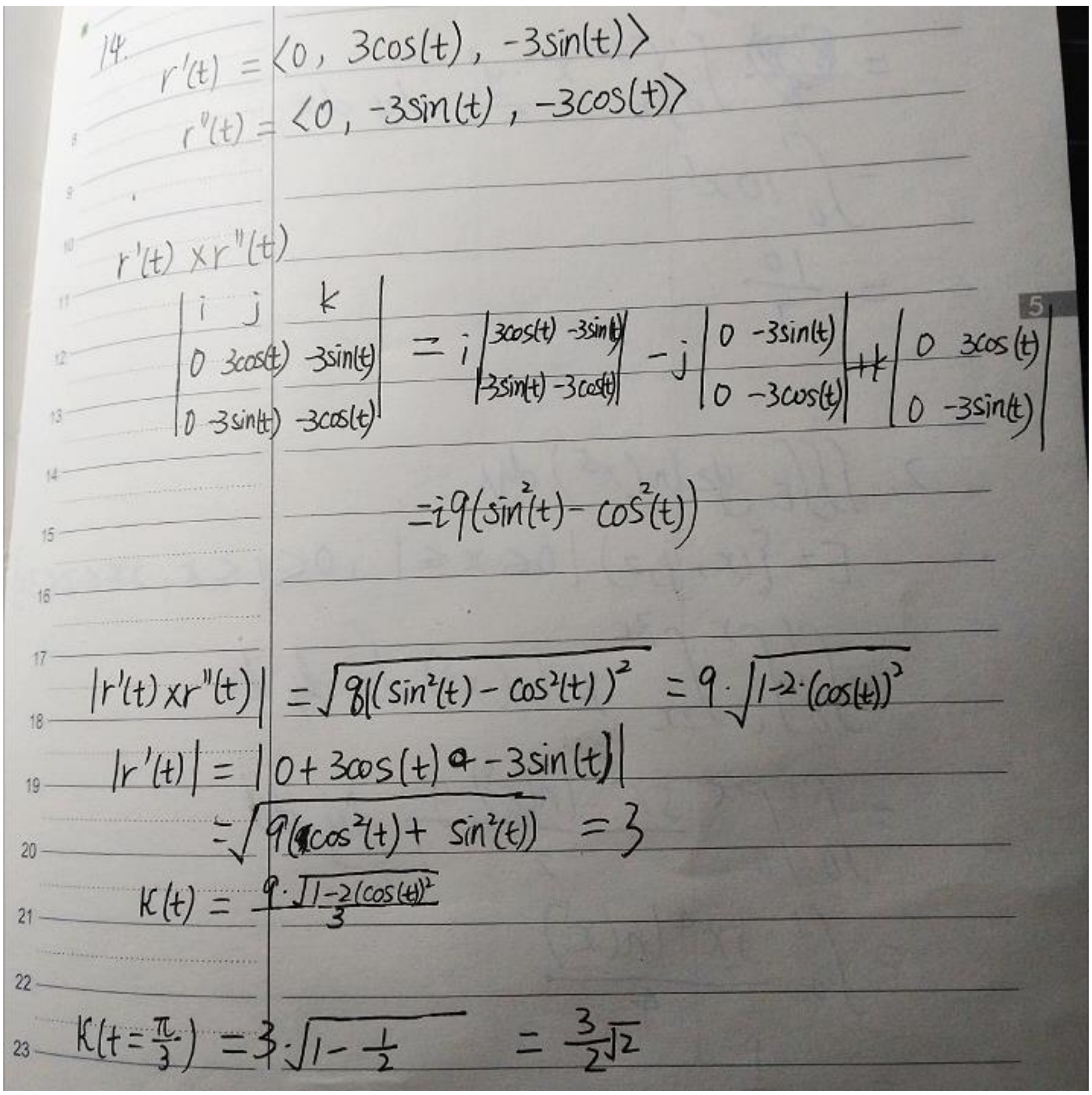
14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = (5, 3\sin t, 3\cos t)$$

at the point where $t = \frac{\pi}{3}$.

---

ans. 3*sqrt(2)/2



14.

$r'(t) = \langle 0, 3\cos(t), -3\sin(t)\rangle$

$r''(t) = \langle 0, -3\sin(t), -3\cos(t)\rangle$

$r'(t) \times r''(t)$

$$\begin{vmatrix} i & j & k \\ 0 & 3\cos(t) & -3\sin(t) \\ 0 & -3\sin(t) & -3\cos(t) \end{vmatrix} = i\begin{vmatrix} 3\cos(t) & -3\sin(t) \\ -3\sin(t) & -3\cos(t) \end{vmatrix} - j\begin{vmatrix} 0 & -3\sin(t) \\ 0 & -3\cos(t) \end{vmatrix} + k\begin{vmatrix} 0 & 3\cos(t) \\ 0 & -3\sin(t) \end{vmatrix}$$

$$= i\,9(\sin^2(t) - \cos^2(t))$$

$$|r'(t) \times r''(t)| = \sqrt{81(\sin^2(t) - \cos^2(t))^2} = 9 \cdot \sqrt{1 - 2\cdot(\cos(t))^2}$$

$$|r'(t)| = |0 + 3\cos(t) - 3\sin(t)|$$

$$= \sqrt{9(\cos^2(t) + \sin^2(t))} = 3$$

$$K(t) = \frac{9 \cdot \sqrt{1 - 2(\cos(t))^2}}{3}$$

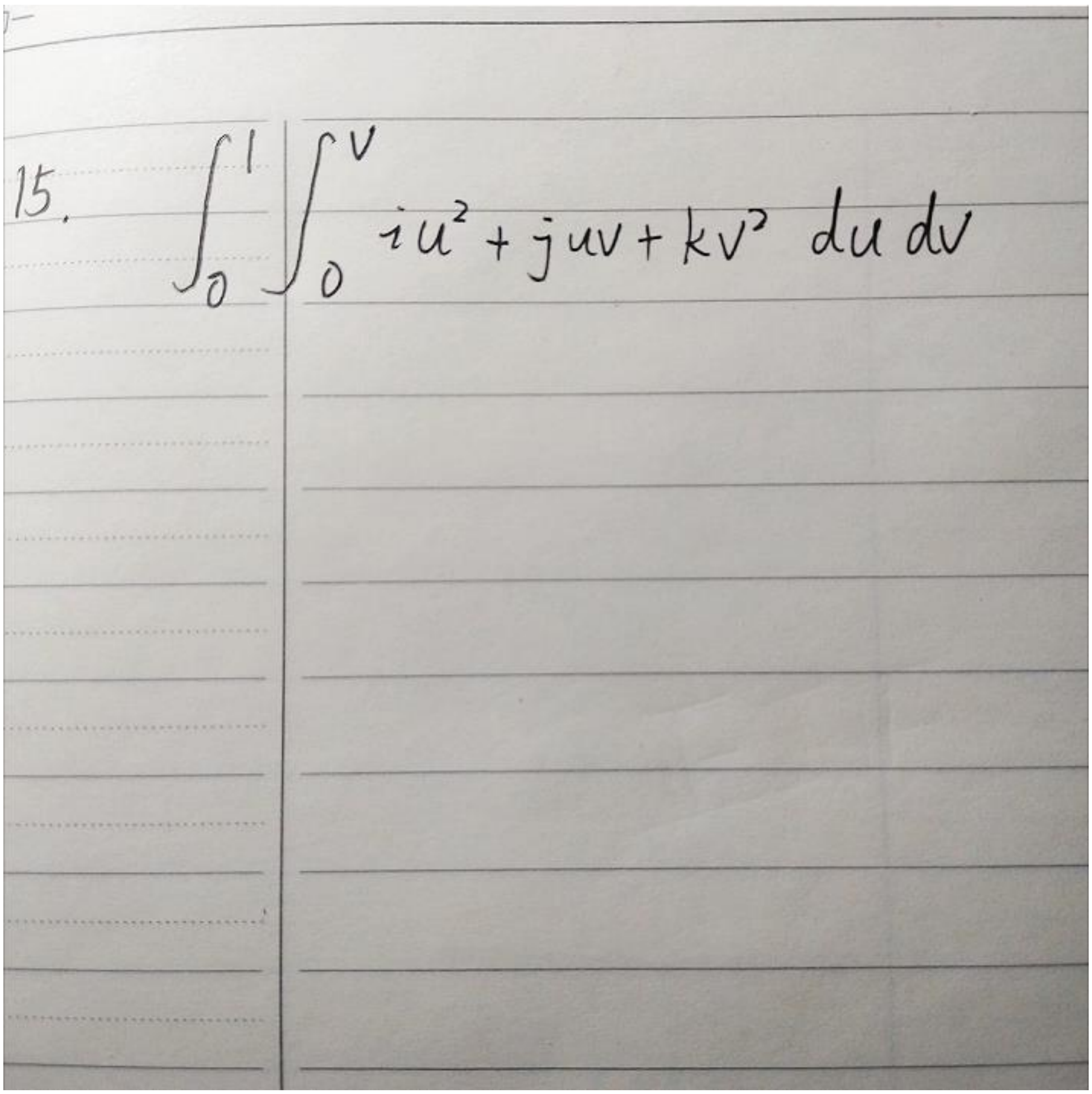
$$K(t = \tfrac{\pi}{3}) = 3 \cdot \sqrt{1 - \tfrac{1}{2}} = \frac{3}{2}\sqrt{2}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u, v) = (u^2\,,\ uv\,,\ v^2) \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. Int(Int(iu^2+juv+kv^2, u=0..v), v=0..1)



16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

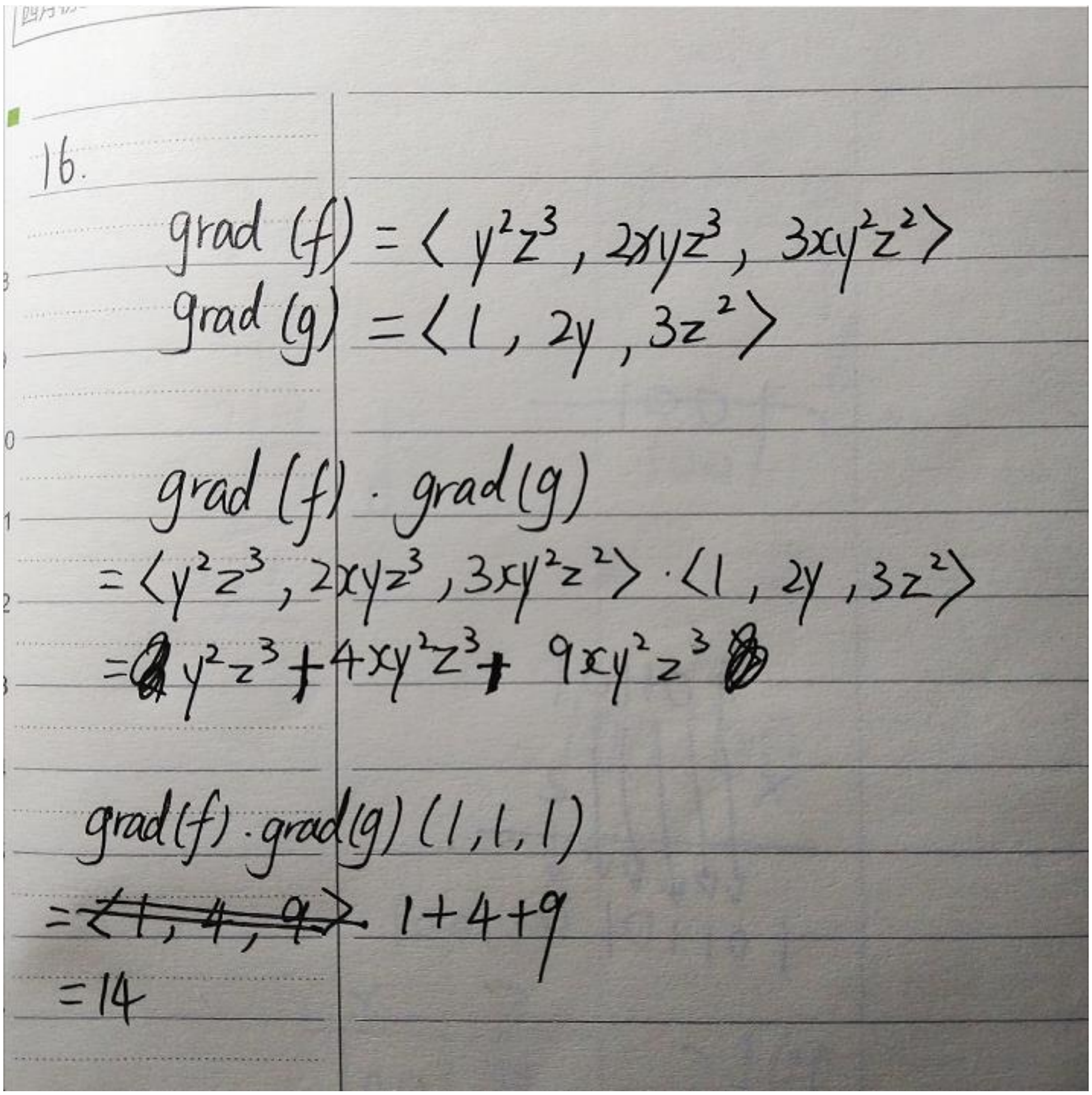
compute the dot-product

$$grad(f) . grad(g) \quad .$$

at the point (1, 1,1).

---

ans. 14



16.

$grad(f) = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$grad(g) = \langle 1, 2y, 3z^2 \rangle$

$grad(f) \cdot grad(g)$

$= \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle \cdot \langle 1, 2y, 3z^2 \rangle$

$= y^2z^3 + 4xy^2z^3 + 9xy^2z^3$

$grad(f) \cdot grad(g) (1,1,1)$

$=$ ~~$\langle 1, 4, 9 \rangle$~~ $1+4+9$

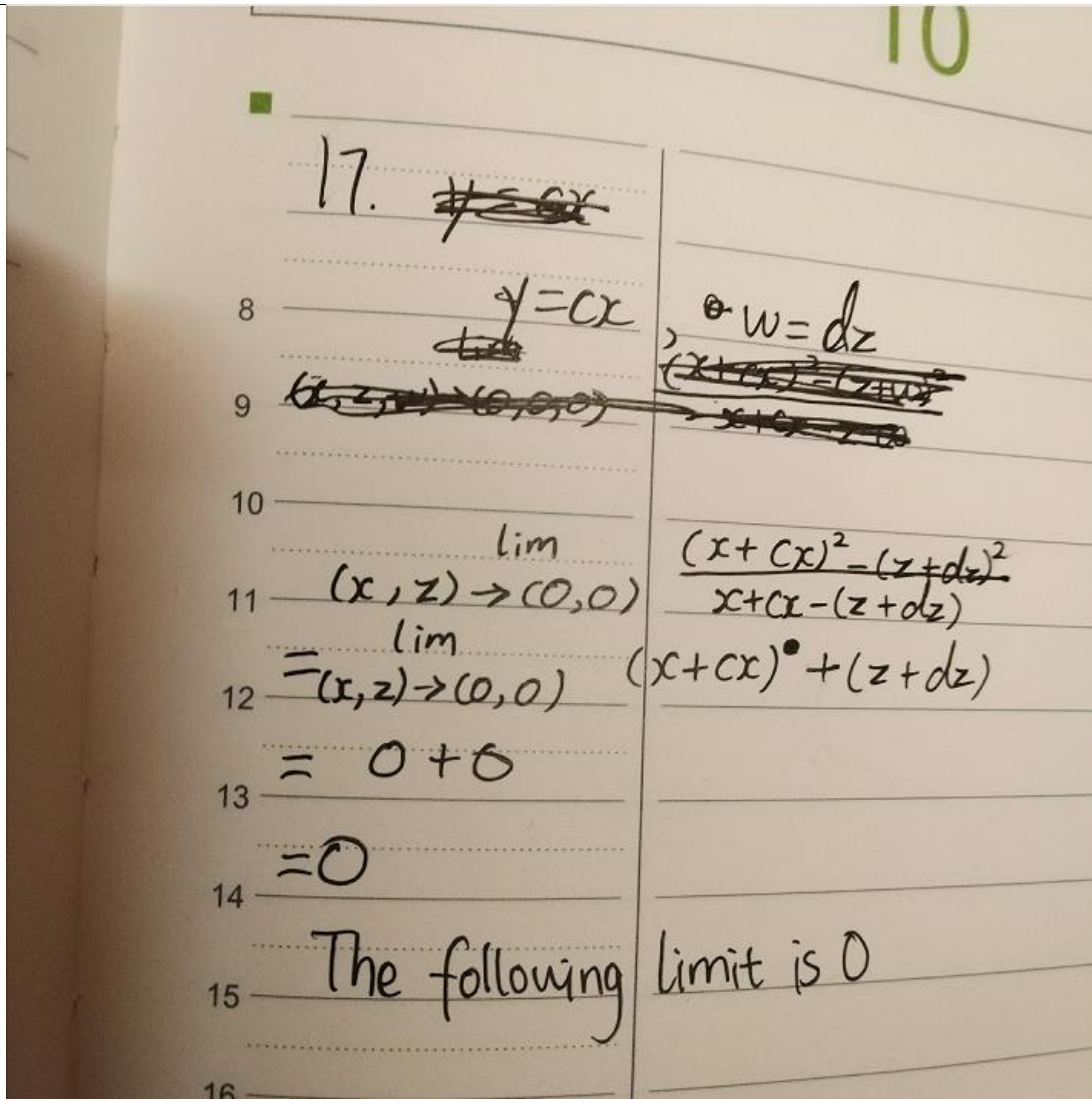
$= 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}\quad .$$

---

ans. 0


17.

$y = cx$, $w = dz$

$$\lim_{(x,z)\to(0,0)} \frac{(x+cx)^2-(z+dz)^2}{x+cx-(z+dz)}$$

$$= \lim_{(x,z)\to(0,0)} (x+cx)+(z+dz)$$

$= 0+0$

$= 0$

The following limit is 0