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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 30
2. $\int_{1/4}^{1}\int_{0}^{\sqrt{y}} f(x,y)\,dx\,dy$
3. ~~π/3, π/3, π/3 for A, B, C respec~~ $-2(x-\pi/6)-10(y-\pi/6)+\pi/6$
4. 0
5. π/3, π/3, π/3 for A, B, C respectively
6. $\cancel{8}$ −24
7. $6\cos(2)$
8. 260π
9. 18
10. Crit pts = (3/4, −1) → local Minimum
11. 3
12. $1/3\sqrt{2}$
13. $\int_0^{\cancel{\pi/4}\,4}\int_0^{\cancel{\pi/4}\,2}\int_0^{2} \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$
14. 1/3
15. $\int_0^1\int_0^v \vec{r}(u,v)\,du\,dv$
16. 14
17. 0

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

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Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the vector-field line integral

$$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

$\vec{r}:(-2,0)\ (0,1)\ (1,1)\ (1,-1)\ (0,1)$

Explain!

---

ans. 30

Use Green's theorem

$P(x,y) = (\cos(e^{\sin x}) + 5y)\cancel{dx}$

$Q(x,y) = (\sin(e^{\cos y}) + 11x)$

$\int_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$\partial Q/\partial x = 11$

$\partial P/\partial y = 5$

$\iint_C (\cancel{11-5}^{\,6})\,dx\,dy$

$12 + 3 + 12 + 3 = 30$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\, \overset{dy\,dx}{\cancel{dx\,dy}}\,.$$

---

ans. $\displaystyle \int_{1/4}^{1} \int_0^{\sqrt{y}} f(x,y)\,dx\,dy$

---

outer integral cannot hav var funcs.
Must be constant

$\sqrt{x} = x^{1/2} = y$

$\cancel{x}$ $y^{1/2} = x$

$x = \sqrt{y}$

$$\int_{1/4}^{1} \int_0^{\cancel{x}=\sqrt{y}} dx\,dy$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -2(x - \pi/6) - 10(y - \pi/6) + \pi/6$

$$L(x,y,z) = f_x(x - \overset{\pi/6}{x_0}) + f_y(y - \overset{\pi/6}{y_0}) + z_0 \to \pi/6$$

$$\frac{dz}{dy} = -2\sin(x+y) + 0 \cdot z'\cancel{(\tfrac{\pi}{6}, \pi/6, \pi/6)}^{x,y,z}$$

$$- 8\sin(y+z) \cdot z'(x,y,z) = 0$$

$$-2\sin(x+y) + z'(x,y,z) - 8\sin(y+z) \cdot z'(x+y,z)$$

$$z' = -2\sin(x+y) - 8\sin(y)$$

$$f_x = -2\cos(x+y) \Rightarrow @\ \pi/6 = -2$$

$$f_y = -2\cos(x+y) - 8\cos(y) \Rightarrow = -10$$

$$L(x,y,z) = -2(x - \pi/6) - 10(y - \pi/6) + \pi/6$$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. 0

$a \times b = [1, 1, -1]$

$b \times c = [1, -1, 1]$ $\quad \therefore$ Ans $= \cancel{0} \times 0$

$a \times c = [2, 1, 2]$

~~$v_1 \; v_3 \; v_3$~~

Given $\vec{v}$ a & b, .. $\vec{a} \times \vec{b}$

$a_2 b_3 - a_3 b_2 = \hat{i} = 1$

$a_3 b_1 - a_1 b_3 = \hat{j} = 1$

$a_1 b_2 - a_2 b_1 = \hat{k} = -1$

5. (12 points) Find the three angles of the triangle ABC where

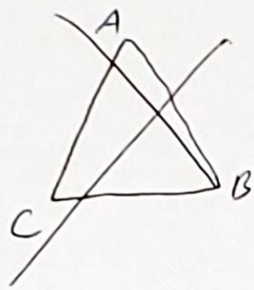
$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\pi/3$ radians ;

The angle at B is: $\pi/3$ radians ;

The angle at C is: $\pi/3$ radians ;



$$\cos\theta = \frac{u \cdot v}{\|u\| \|v\|}$$

$$\cos^{-1}\left(\frac{u \cdot v}{\|u\| \|v\|}\right) = \theta$$

$$\vec{r}_{AB} = [1, 0, 1] \rightarrow |\vec{r}_{AB}| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$$\vec{r}_{BC} = [0, 1, -1] \rightarrow |\vec{r}_{\cancel{AB}BC}| = \sqrt{2}$$

$$\vec{r}_{AC} = [1, 1, 0] \rightarrow |\vec{r}_{AC}| = \sqrt{2}$$

$$\theta_A = \cos^{-1}\left(\frac{1(1) + 1(0) + 1(0)}{\sqrt{2} \cdot \sqrt{2}}\right) = 60 = \pi/3 \qquad \theta_C = \cos^{-1}\left(\frac{1(0) + 1(1) + -1(0)}{\sqrt{2} \cdot \sqrt{2}}\right)$$

$$\theta_B = \cos^{-1}\left(\frac{1(0) + 0(1) + 1(-1)}{\sqrt{2} \cdot \sqrt{2}}\right) = \overset{60}{\cancel{120}} = \cancel{2\pi/3}\ \pi/3 \qquad = 60 = \pi/3$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans. -24

$$\nabla f = \begin{bmatrix} 3x^2 + yz \\ 3y^2 + xz \\ 3z^2 + xy \end{bmatrix} @ (1,1,1) = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$$

$A = (1,1,1)$

$B = (-1,-1,-1)$

$\vec{r}_{AB} = [-2, -2, -2] = \vec{v}$

$$D_{\vec{v}} f = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} = -24$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad .$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos(2)$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial x} = 6x \qquad \frac{\partial g}{\partial y} = -6y$$

$$\frac{\partial x}{\partial u} = \cancel{\cos(v) e^u}$$
$$\cancel{e^u \cos v}$$
$$e^u \cos v$$

$$\frac{\partial y}{\partial u} = \cancel{e^u \cos(v)}$$
$$\cancel{e^u \sin v}$$
$$e^u \sin v$$

$$\cancel{6x(e^u \cos}$$

$$6(e^u \cos v)(e^u \cos v) + -6(e^u \sin v)(e^u \sin v) @ (0,1) = (u,v)$$

$$= 6\cos(2) = -2.4969$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3x + \cos(y^3 + yz),\ -2y + e^{x+z^2},\ 5z + \sin(xy^3 + e^x) \rangle \quad .$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 260π

$$\iint_S F \cdot dS$$

$$\int_0^1 \int_0^x$$

$$\vec{r}_\theta = \cancel{x}\,2 \sin\phi\, \cancel{\cos}\,\sin\theta + \cancel{y}\,2 \sin\phi\, \cancel{\sin}\,\cos\theta\ \cancel{+ z\,2\cos\phi}$$

$$\vec{r}_\phi = 2\cos\phi\cos\theta + 2\cos\phi\sin\theta - 2\sin\phi$$

$$\vec{r}_\theta \times \vec{r}_\phi = 4\sin^2\phi\cos\theta - 4\sin^2\phi\sin\theta - 4\sin\phi\cos\phi = \vec{n}$$

$$\vec{n} \cdot F = 260\pi$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. 18

$$\nabla f = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad |\nabla f| = 1$$

$$\iint_S \vec{F} \cdot \vec{n}\, dS$$

$$\iint_S \left[F \cdot \frac{\partial z}{\partial x} \hat{i}, \frac{\partial z}{\partial y} \hat{j} \right] dx\,dy$$

$$z_x = 2 \qquad |z| = \sqrt{13}$$

$$z_y = 3$$

$$n = \frac{z_x \times z_y}{|z_x \times z_y|} = \frac{[0\ 0\ 6]}{6}$$

$$\int_0^1 \int_0^1 \overset{3z,\,2x,\,y+z}{[\cancel{0\ 0\ 1}]} \cdot \frac{[0\ 0\ 6]}{6}\ \underbrace{dx\,dy}_{dS} = 18$$

$$\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y)$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. Crit pts = $(+3/4, -1) \to$ local Minimum

$$f(x,y) = 4x - y^2 - \ln(2x+y)$$

$$\partial f/\partial x = -\frac{2}{2x+y} + 4 = 0$$

$$\partial f/\partial y = -2y - \frac{1}{2x+y} = 0$$

$x = \frac{3}{4}$, $y = -1$ make both eqs $= 0$

$\left(\frac{3}{4}, -1\right)$ = critical pts

$$\frac{\partial^2 f}{\partial x^2} = 4/(2x+y)^2 \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{2}{(2x+y)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = -2 + \frac{1}{(2x+y)^2} \qquad \frac{4}{(2 \cdot \frac{3}{4} + (-1))^2}\left(-2 + \frac{1}{(2 \cdot \frac{3}{4} - 1)^2}\right) -$$

$$\frac{2}{(2(\frac{3}{4})-1)^2} = 24 > 0$$

max/min

f_{xx} & $f_{yy} = +$,

pos ~~neg~~ concavity minimum

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3

$$L_f(x, y, z) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + f_z(x_0, y_0)(z - z_0) + f(x_0, y_0, z_0)$$

$$f_x = 2x \Big/ \sqrt{2x^2 + 3y^2 + z^2} \rightarrow @(1,1,2) = 2/3$$

$$f_y = 3y \Big/ \sqrt{2x^2 + 3y^2 + z^2} \rightarrow 1$$

$$f_z = z \Big/ \sqrt{2x^2 + 3y^2 + z^2} \rightarrow 2/3$$

$$= 2/3(x - 1) + 1(y - 1) + 2/3(z - 2) + 3$$

$$= 2/3\,x - 2/3 + y - 1 + 2/3\,z - \tfrac{4}{3} + 3$$

$$= \frac{2x + 2z}{3} + y \rightarrow \frac{2(1.001) + 2(2.001)}{3} + (0.999)$$

$$= 3.000333\ldots$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $1/3\sqrt{2}$

$$\left.\int_0^{45}\int_0^{1} \quad + \int_{45}^{1}\int_0^{1}\right\}$$ convert to polar coords

$\cancel{x=}\ x = (1-y)^2$

$$\int_0^{45}\underbrace{\int_0^{1}(1-y)^2\,dy}\,dx$$

$\hookrightarrow = 1/3 \to \int_0^{45} 1/3\,dx = .15$

$.15 + \int_{45}^{1} 1/3\,dx$

$= 44/3\sqrt{2}$

$\hookrightarrow 5 - \frac{44}{3\sqrt{2}} = \frac{1}{3\sqrt{2}}$

13. (12 points) Convert the triple iterated integral

$$\int_0^4 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^0 x^2\, y\, z\, dx\, dy\, dz$$

to spherical coordinates. **Do not evaluate.**

---

ans. $$\int_0^{\overset{4}{\cancel{\pi/4}}} \int_0^{\overset{2}{\cancel{\pi/4}}} \int_0^{\cancel{4}\,2} \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

---

convert to ρ, ϕ, θ

$$\rho = \sqrt{x^2+y^2+z^2}$$

$$1\, dx\, dy\, dz = \underbrace{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$$

$\cancel{\rho}$ $x=1$ $y=1$ $z=1$

$\rho = 1$

$\phi = \arctan(1/1) = 45^\circ$

$\theta = 45^\circ$

Use Jacobian determinant

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $1/3$

$$K(t) = \frac{\|\vec{r}\,'(t) \times \vec{r}\,''(t)\|}{\left(\|\vec{r}\,'(t)\|\right)^3}$$

$$\vec{r}\,(t) = [0, 3\cos(t), -3\sin(t)]$$

$$\vec{r}\,''(t) = [0, -3\sin(t), -3\cos(t)]$$

$$|\vec{r}\,'(t)| = \sqrt{(0)^2 + (3\cos t)^2 + (-3\sin t)^2} = 3$$

$$\hookrightarrow 3^3 = 27$$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = [-9, 0, 0]$$

$$\sqrt{(-9)^2 + 0^2 + 0^2} = 9$$

$$\frac{9}{27} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 , uv , v^2 \rangle \quad , \quad 0 < u < v < 1 \ .$$

---

ans. $\int_0^1 \int_0^v \vec{r}(u,v)\, du\, dv$

$\cancel{\int_0^1 \int_0^u r(u,v)\, du\, dv}$

$0 < u < v < 1$

↳ Tells us domain for Bounds of integration

$\int_0^1 \int_0^v \vec{r}(u,v)\, du\, dv$,

where $\vec{r}(u,v) = [u^2, uv, v^2]$

Answer

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g)$$

at the point $(1,1,1)$.

---

ans. 14

$$\nabla f = \begin{bmatrix} y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{bmatrix} \qquad \nabla g = \begin{bmatrix} 1 \\ 2y \\ 3z^2 \end{bmatrix}$$

$$@(1,1,1) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \qquad \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$1(1) + 2(2) + 3(3) = 14$$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

---

ans. 0

$\cancel{(x+y)(x+y)}$

B/C Diff of squares

$$\frac{\cancel{((x+y)-(z+w))}((x+y)+(z+w))}{\cancel{x+y-z-w}}$$

$x+y+z+w = 0+0+0+0 = 0$

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