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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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Sign the following declaration:

I Shaun Goda Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Shaun G

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

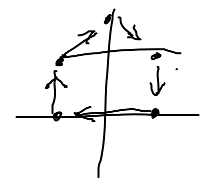
$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y})\;+\;11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$



clockwise

Explain!

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ans. -18

$$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy$$

$$= -\iint_D (11 - 5)\,dA = -\left(\int_{-1}^{1}\int_0^1 6\,dy\,dx + \int_{-1}^{0}\int_1^{x+2} 6\,dy\,dx + \int_0^1\int_1^{-x+2} 6\,dy\,dx\right)$$

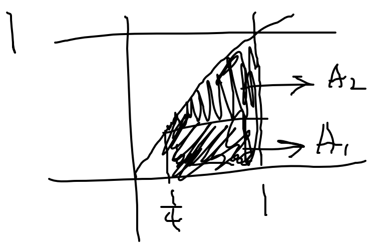
$$\int_{-1}^{1}\int_0^1 6\,dy\,dx = 12 \qquad \int_{-1}^{0}\int_1^{x+2} 6\,dy\,dx = 3 \qquad \int_0^1\int_1^{-x+2} 6\,dy\,dx = 3$$

$$-(12+3+3) = -18$$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx \quad .$$

ans. $\displaystyle\int_{0}^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$



3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -3\sqrt{3}x - 5\sqrt{3}y - 6\sqrt{3}z + \frac{\pi(14\sqrt{3}+1)}{6}$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z)$

$f_x = -2\sin(x+y) - 4\sin(x+z)$

$f_y = -2\sin(x+y) - 8\sin(y+z)$

$f_z = -4\sin(x+z) - 8\sin(y+z)$

$f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -3\sqrt{3}$

$f_y(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -5\sqrt{3}$

$f_z(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -6\sqrt{3}$

$z - \frac{\pi}{6} = -3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6})$

$z = -3\sqrt{3}x + \frac{\pi\sqrt{3}}{2} - 5\sqrt{3}y + \frac{5\pi\sqrt{3}}{6} - 6\sqrt{3}z + \pi\sqrt{3} + \frac{\pi}{6}$

$z = -3\sqrt{3}x - 5\sqrt{3}y - 6\sqrt{3}z + \frac{\pi(14\sqrt{3}+1)}{6}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

$\langle 1, 1, -1 \rangle$ $\quad \langle 1, -1, 1 \rangle$ $\quad \langle 2, 1, 2 \rangle$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans.

$$\begin{vmatrix} i & j & k \\ a & b & c \\ 2a & -b & 3c \end{vmatrix} =$$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \qquad .$$

$\arcsin\left(\frac{\sqrt{3}}{3}\right) \approx 0.6155$

ans. The angle at A is: radians ;

The angle at B is: $\frac{2\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{2}$ radians ;

$\vec{AB} = \langle 1, 0, 1 \rangle \quad \vec{AC} = \langle 1, 1, 1 \rangle \quad \vec{BC} = \langle 0, 1, -1 \rangle$

$\|\vec{AB}\| = \sqrt{2} \quad \|\vec{AC}\| = \sqrt{3} \quad \|\vec{BC}\| = \sqrt{2}$

$\vec{AB}, \vec{AC} \quad \arccos\left(\frac{1+0+1}{\sqrt{2}\cdot\sqrt{3}}\right) = 0.615$

$\vec{AB}, \vec{BC} \quad \arccos\left(\frac{0+0-1}{\sqrt{2}\cdot\sqrt{2}}\right) = \frac{2\pi}{3}$

$\vec{AC}, \vec{BC} \quad \arccos\left(\frac{0+1-1}{\sqrt{3}\cdot\sqrt{2}}\right) = \frac{\pi}{2}$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

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ans. $-4\sqrt{3}$

$f_x = 3x^2 + yz \quad f_y = 3y^2 + xz \quad f_z = 3z^2 + xy$

$|\langle -1,-1,-1\rangle| = \sqrt{(-1)^2+(-1)^2+(-1)^2} = \sqrt{3}$

$u = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$\nabla f(1,1,1) = \langle 3+1, 3+1, 3+1 \rangle = \langle 4, 4, 4 \rangle$

$\langle 4, 4, 4 \rangle \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle = 3(4)(-\frac{1}{\sqrt{3}})$

$= -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

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ans. 6

$$g(u,v) = 3e^{2u}\cos^2 v - 3e^{2u}\sin^2 v$$

$$\frac{\partial}{\partial u} g(u,v) = 6e^{2u}\cos^2 v - 6e^{2u}\sin^2 v$$

$$\frac{\partial}{\partial u} g(0,1) = 6(1)(1) - 6(1)(0) = 6$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^{2} 6\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$\mathrm{div}\, F = 3 - 2 + 5 = 6$$

$$\int_0^2 6\rho^2 \sin\phi \, d\rho = \left| 2\rho^3 \sin\phi \right|_0^2 = 16 \sin\phi$$

$$\int_0^{\frac{\pi}{2}} 16 \sin\phi \, d\theta = 8\pi \sin\phi$$

$$\int_0^{\frac{\pi}{2}} 8\pi \sin\phi \, d\phi = \left| -8\pi \cos\phi \right|_0^{\frac{\pi}{2}} = 8\pi$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

$$\iint_S F.dS = \iint_D \Big(-\big(3(2x+3y)\big)(2) - (2x)(3) + \big(y + (2x+3y)\big)\Big)\, dA$$

$z_x = 2,$

$z_y = 3$

$$= \int_0^1 \int_0^1 (-16x - 14y)\, dy\, dx$$

$$\Rightarrow \int_0^1 (-16x - 14y)\, dy = \Big| -16xy - 7y^2 \Big|_0^1 = -16x - 7$$

$$\int_0^1 (-16x - 7)\, dx = \Big| -8x^2 - 7x \Big|_0^1 = -8 - 7$$

$$= -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. Saddle point @ $\left(-2, \frac{5}{4}\right)$

$f_x = 4 - \frac{2}{2x+y}$ $\quad f_y = -2y - \frac{1}{2x+y}$ $\quad f_{xy} = \frac{2}{(2x+y)^2}$

$f_{xx} = \frac{4}{(2x+y)^2}$ $\quad f_{yy} = -2 + \frac{1}{(2x+y)^2}$

$0 = 4 - \frac{2}{2x+y}$ $\quad 0 = -2y - \frac{1}{2x+y}$

$\frac{2}{2x+y} = 4$ $\quad 0 = -2y - 4$

$4 = -2y$

$y = -2$

$0 = 4 - \frac{2}{2x-2}$ $\quad \left(-2, \frac{5}{4}\right)$

$\frac{2}{2x-2} = 4$ $\quad f_{xx}\left(-2, \frac{5}{4}\right) = \frac{64}{121}$

$2 = 8x - 8$ $\quad f_{xy}\left(-2, \frac{5}{4}\right) = \frac{32}{121}$

$10 = 8x$ $\quad f_{yy}\left(-2, \frac{5}{4}\right) = \frac{-226}{121}$

$x = \frac{5}{4}$

$D = \left(\frac{64}{121}\right)\left(\frac{-226}{121}\right) - \left(\frac{32}{121}\right)^2 = -\frac{128}{121}$

$D < 0$ saddle

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) \;=\; \sqrt{2x^2+3y^2+z^2} \quad .$$

$$= (2x^2+3y^2+z^2)^{\frac{1}{2}}$$

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ans. $3.000\dot{3}$

$$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} \quad f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} \quad f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

$$f(1,1,2) = 3$$

$$f_x(1,1,2) = \frac{2}{3} \quad f_z(1,1,2) = \frac{2}{3}$$

$$f_y(1,1,2) = 1$$

$$L(x,y) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$$

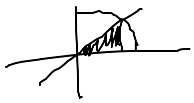
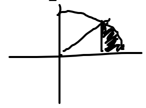
$$= 3 + \frac{2}{3}x - \frac{2}{3} + y - 1 + \frac{2}{3}z - \frac{4}{3}$$

$$f(x,y) \approx \frac{2}{3}x + y + \frac{2}{3}z$$

$$f(1.001, 0.999, 2.001) = 3.000\dot{3}$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx \; + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain!  +  $\frac{\pi}{4}$ degree

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ans. $\frac{\sqrt{2}}{6}$

$r = 1, \quad x = r\cos\theta$

$$\int_0^{\frac{\pi}{4}} \int_0^1 r^2 \cos\theta \, dr \, d\theta \Rightarrow \int_0^1 r^2 \cos\theta \, dr = \left. \frac{r^3}{3} \cos\theta \right|_0^1 = \frac{1}{3} \cos\theta$$

$$\frac{1}{3} \int_0^{\frac{\pi}{4}} \cos\theta \, d\theta = \frac{1}{3} \left. \sin\theta \right|_0^{\frac{\pi}{4}} = \frac{1}{3}\left(\frac{\sqrt{2}}{2} - 0\right)$$

$$= \frac{\sqrt{2}}{6}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^4 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

$x \le 0$
$y \ge 0$
$z \ge 0$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $$\int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2} \rho^6 \sin^4\phi \cos\phi \sin\theta \, d\rho\, d\phi\, d\theta$$

$x = \rho\sin\phi\cos\theta \quad y = \rho\sin\phi\sin\theta \quad z = \rho\cos\phi$

$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 y z \, dx\, dy\, dz = \iiint_V (\rho^2\sin^2\phi\cos^2\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\, dV$$

$$= \iiint_V (\rho^6 \sin^4\phi \cos\phi \sin\theta)\, dV$$

$$= \int_{\frac{\pi}{2}}^{\pi} \int_0^{\frac{\pi}{2}} \int_0^{2} \rho^6 \sin^4\phi \cos\phi \sin\theta \, d\rho\, d\phi\, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

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ans. $K(t) = \frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix}$$

$$= (-9\cos^2 t - 9\sin^2 t)i - (0-0)j + (0-0)k$$

$$= \langle -9, 0, 0\rangle$$

$\|r'(t) \times r''(t)\| = \sqrt{81} = 9$

$\|r'(t)\| = \sqrt{0^2 + 9\cos^2 t + 9\sin^2 t} = \sqrt{9} = 3$

$$K(t) = \frac{9}{3^3} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) \;=\; \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \quad .$$

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ans. $\displaystyle\int_v^1 \int_0^u \sqrt{4(u^2+v^2)^2}\, dv\, du$

$$\text{Area} = \iint_D \| r_u \times r_v \| \, dA$$

$$r_u = \langle 2u, v, 0 \rangle \qquad r_u \times r_v = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix}$$

$$r_v = \langle 0, u, 2v \rangle$$

$$= (2v^2 - 0)\, i - (4uv - 0)\, j + (2u^2 - 0)\, k$$

$$= \langle 2v^2, -4uv, 2u^2 \rangle$$

$$\| r_u \times r_v \| = \sqrt{4v^4 + 16u^2v^2 + 4u^2} = \sqrt{4(u^2+v^2)^2}$$

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \qquad .$$

at the point $(1,1,1)$.

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ans. 14

$$\nabla f = \langle y^2 z^3, \; 2xyz^3, \; 3xy^2z^2 \rangle$$

$$\nabla g = \langle 1, \; 2y, \; 3z^2 \rangle$$

$$\nabla f \cdot \nabla g = y^2z^3 + 4xy^2z^3 + 9xy^2z^4$$

$$@(1,1,1) = 1 + 4 + 9 = 14$$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. 0

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{0}{0} \qquad w = 2x$$

$$\frac{(x+y^2)-(z+w)^2}{x+y-z-w} = \frac{(x+y+z+w)\cancel{(x+y-z-w)}}{\cancel{(x+y-z-w)}}$$

$$= x+y+z+w$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} x+y+z+w = 0$$