NAME: (print!) Yeram Sarah Jung **RUID:** (print!) 192004589

SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

---

1. 0

2. $\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$

3. $z = -0.5x - \frac{5}{6}y + \frac{7\pi}{18}$

4.

5. all are $\pi/3$ rad

6. $-4\sqrt{3}$

7. $6\cos(1)x - 6\sin(1)y$

8. 8π

9. 15

10. $(-\frac{1}{4}, 1)$; saddle point

11. 3.00033

12. $\frac{\sqrt{2}}{4}$

13. $\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,d\rho\,d\phi\,d\theta$

14. $\frac{1}{81}$

15.

16. 14

17. 0

Sign the following declaration:

I Yeram Sarah Jung Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Sarah Jung

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\, dx + (sin\,(e^{\cos y}) + 11x\,)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans. 0

$P = \cos(e^{\sin x}) + 5y \qquad Q = \sin(e^{\cos y}) + 11x$

$\frac{dQ}{dx} - \frac{dP}{dy} = 11 - 5 = 6$

$\iint_D 6\, dA$

$D = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 2x\}$

$\int_{-1}^{1}\int_{0}^{2x} 6\, dy\, dx$

$= \int_{-1}^{1} 6y \Big|_0^{2x} dx = \int_{-1}^{1} 12x\, dx = 6x^2 \Big|_{-1}^{1} = 6(1)^2 - 6(-1)^2 = \boxed{0}$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x, y) \overset{dy\ dx}{\cancel{dx\,dy}} .$$

---

ans. $\displaystyle\int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\ dx\,dy$

$D = \{(x,y) \mid \frac{1}{4} \le x \le 1,\ 0 \le y \le \sqrt{x}\}$

$y^2 \le x \le 1 \quad \frac{1}{2} \le y \le 1$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -0.5x - \frac{5}{6}y + \frac{7\pi}{18}$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x(x,y,z) = -2\sin(x+y) - 4\sin(x+z)$

$f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -\sqrt{3} - 2\sqrt{3} = -3\sqrt{3}$

$f_y(x,y,z) = -2\sin(x+y) - 8\sin(y+z)$

$f_y(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -\sqrt{3} - 4\sqrt{3} = -5\sqrt{3}$

$f_z(x,y,z) = -4\sin(x+z) - 8\sin(y+z)$

$f_z(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -2\sqrt{3} - 4\sqrt{3} = -6\sqrt{3}$

Tangent Plane Eq

$(x - \frac{\pi}{6})(-3\sqrt{3}) + (y - \frac{\pi}{6})(-5\sqrt{3}) + (z - \frac{\pi}{6})(-6\sqrt{3}) = 0$

$-3\sqrt{3}x + \frac{\sqrt{3}\pi}{2} - 5\sqrt{3}y + \frac{5\sqrt{3}\pi}{6} - 6\sqrt{3}z + \sqrt{3}\pi = 0$

$-3\sqrt{3}x - 5\sqrt{3}y - 6\sqrt{3}z + \frac{14\sqrt{3}\pi}{6} = 0$

$6\sqrt{3}z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{7\sqrt{3}\pi}{3}$

$z = -0.5x - \frac{5}{6}y + \frac{7\pi}{18}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans.

---

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\pi/3$ radians ;

The angle at B is: $\pi/3$ radians ;

The angle at C is: $\pi/3$ radians ;

$|AB| = \sqrt{(0-1)^2+(0-0)^2+(0-1)^2} = \sqrt{2}$

$|BC| = \sqrt{(1-1)^2+(0-1)^2+(1-0)^2} = \sqrt{2}$

$|CA| = \sqrt{(1-0)^2+(1-0)^2+(0-0)^2} = \sqrt{2}$

$|AB| = |BC| = |CA| = \sqrt{2}$

$\hookrightarrow$ equilateral all equal $60^\circ = \frac{\pi}{3}$ rad

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans. $-4\sqrt{3}$

$f(x,y,z) = x^3 + y^3 + z^3 + xyz$

$\nabla f(x,y,z) = \langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \rangle = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle$

$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$

$V = \langle -1-1, -1-1, -1-1 \rangle = \langle -2, -2, -2 \rangle$

$u = \dfrac{\langle -2, -2, -2 \rangle}{\sqrt{(-2)^2 + (-2)^2 + (-2)^2}} = \langle \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}} \rangle = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$D_u f(1,1,1) = \nabla f(1,1,1) \cdot u$

$= \langle 4,4,4 \rangle \cdot \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$

$= -\frac{4}{\sqrt{3}} + (-\frac{4}{\sqrt{3}}) + (-\frac{4}{\sqrt{3}}) = \frac{-12}{\sqrt{3}} = \frac{-12\sqrt{3}}{3} = \boxed{-4\sqrt{3}}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos(1)\,x - 6\sin(1)\,y$

$$\frac{dg}{du} = \frac{dg}{dx}\frac{dx}{du} + \frac{dg}{dy}\frac{dy}{du}$$

$$\frac{dg}{dx} = 6x \qquad \frac{dx}{du} = e^u \cos v$$

$$\frac{dg}{dy} = -6y \qquad \frac{dy}{du} = e^u \sin v$$

$$\frac{dg}{du} = (6x)(e^u \cos v) + (-6y)(e^u \sin v)$$

$$\frac{dg}{du} = (6x)(e^0 \cos(1)) + (-6y)(e^0 \sin(1))$$

$$\frac{dg}{du} = 6\cos(1)\,x - 6\sin(1)\,y$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3x + \cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\rangle \qquad ,$$

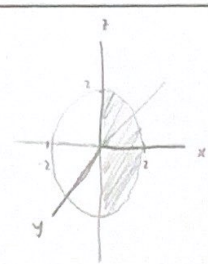
and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

$\operatorname{div} F = 3 - 2 + 5 = 6$



$\iint_S F \cdot ds = \iiint_E 6 \, dV$

$r = 2 \quad \phi = \frac{\pi}{2} \quad \theta = \frac{\pi}{2}$

$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} 6\rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

$= 2\rho^3 \sin\phi \Big|_0^2 = 16\sin\phi$

$\int_0^{\pi/2}\int_0^{\pi/2} 16\sin\phi \, d\theta \, d\phi$

$= 16\theta \sin\phi \Big|_0^{\pi/2} = 8\pi \sin\phi$

$\int_0^{\pi/2} 8\pi \sin\phi \, d\phi$

$= -8\pi\cos\phi \Big|_0^{\pi/2} = 0 - (-8\pi) = \boxed{8\pi}$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\,2x\,,\,y+z\rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. 15

$P = 3z \qquad Q = 2x \qquad R = y + z$

$\frac{dz}{dx} = 2 \qquad \frac{dz}{dy} = 3$

$\iint (-(3z)(2) - (2x)(3) + y + z)\, dA$

$= \iint (-6z - 6x + y + z)\, dA = \iint (-6x + y - 5z)\, dA$

$= \iint (-6x + y - 5(2x + 3y))\, dA = \iint (-6x + y - 10x - 15y)\, dA = \iint (-16x - 14y)\, dA$

$\int_0^1 \int_0^1 (-16x - 14y)\, dy\, dx$

$= \int_0^1 -16xy - 7y^2 \Big|_0^1 dx = \int_0^1 (-16x - 7)\, dx = -8x^2 - 7x \Big|_0^1 = -8 - 7 = -15$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. $(-\frac{1}{4}, 1)$; saddle point

$f_x = 4 - \frac{2}{2x+y}$ $\qquad f_y = -2y - \frac{1}{2x+y}$

$f_{xx} = \frac{4}{(2x+y)^2}$ $\qquad f_{xy} = \frac{2}{(2x+y)^2}$ $\qquad f_{yy} = \frac{1}{(2x+y)^2} - 2$

$4 - \frac{2}{2x+y} = 0$ $\qquad -2y - \frac{1}{2x+y} = 0$

$4(2x+y) - 2 = 0$

$2x + y = \frac{1}{2}$

$y = \frac{1}{2} - 2x$

$-2(\frac{1}{2} - 2x) - \frac{1}{2x + (\frac{1}{2} - 2x)} = 0$

$= -1 + 4x - \frac{1}{-\frac{1}{2}}$

$= 4x + 1 = 0$

$4x = -1$

$x = -\frac{1}{4}$ $\quad y = 1$

critical point: $(-\frac{1}{4}, 1)$

$f_{xx}(-\frac{1}{4}, 1) = \frac{4}{2(-\frac{1}{4}) + 1} = 8$

$f_{xy}(-\frac{1}{4}, 1) = \frac{2}{(2(-\frac{1}{4})+1)^2} = 8$ $\qquad D = (8)(4) - 8^2 = -32 \leftarrow$ saddle point

$f_{yy}(-\frac{1}{4}, 1) = \frac{1}{(2(-\frac{1}{4})+1)^2} = 4$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.00033

$L(x,y,z) = f(1,1,2) + f_x(1,1,2)(x-1) + f_y(1,1,2)(y-1) + f_z(1,1,2)(z-2)$

$f_x(x,y,z) = \dfrac{2x}{\sqrt{2x^2+3y^2+z^2}}$ $f(1,1,2) = 3$

$f_y(x,y,z) = \dfrac{3y}{\sqrt{2x^2+3y^2+z^2}}$ $f_x(1,1,2) = \frac{2}{3}$

$f_y(1,1,2) = 1$

$f_z(x,y,z) = \dfrac{z}{\sqrt{2x^2+3y^2+z^2}}$ $f_z(1,1,2) = \frac{2}{3}$

$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$

$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(0.001) + 1(-0.001) + \frac{2}{3}(0.001)$

$= \boxed{3.00033}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $\frac{\sqrt{2}}{4}$

$y = x$

$x = r\cos\theta$

$y = \sqrt{1-x^2}$

$y^2 = 1 - x^2$

$x^2 + y^2 = 1 \qquad r = 1$

$\frac{\sqrt{2}}{2} = \cos\theta$

$\theta = 45° = \frac{\pi}{4}$

$$\int_0^{\pi/4} \int_0^1 r\cos\theta\,dr\,d\theta$$

$$= \frac{r^2}{2}\cos\theta \Big|_0^1 = \frac{\cos\theta}{2}$$

$$\int_0^{\frac{\pi}{4}} \frac{\cos\theta}{2}\,d\theta = \frac{\sin\theta}{2}\Big|_0^{\pi/4} = \frac{\sqrt{2}}{4} - 0 = \boxed{\frac{\sqrt{2}}{4}}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to spherical coordinates. Do not evaluate.

---

ans. $\int_0^{\pi/2}\int_0^{2\pi}\int_0^{2} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\, d\rho\, d\phi\, d\theta$

$x = \rho\sin\phi\cos\theta$ $\quad y = \rho\sin\phi\sin\theta$ $\quad z = \rho\cos\phi$ $\qquad x^2+y^2+z^2=\rho$

$x^2 = 4 - z^2 - y^2$

$\rho = 2$

$z = 2\cos\phi$

$y = \sqrt{4-z^2}$ $\qquad 2 = 2\cos\phi$

$y^2 = 4 - z^2$ $\qquad 1 = \cos\phi$

$\phi = 2\pi$

$x^2 = 4 - z^2 - 4 + z^2$

$x^2 = 0$

$x = 0$ $\qquad = \int_0^{\pi/2}\int_0^{2\pi}\int_0^{2} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\, d\rho\, d\phi\, d\theta$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{1}{81}$

$r'(t) = 0 + 3\cos t\, j - 3\sin t\, k$ $\qquad r'(\frac{\pi}{3}) = \langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle$

$r''(t) = 0 + (-3\sin t)\, j - 3\cos t\, k$ $\qquad r''(\frac{\pi}{3}) = \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & 3/2 & -\frac{3\sqrt{3}}{2} \\ 0 & -\frac{3\sqrt{3}}{2} & -\frac{3}{2} \end{vmatrix} = i\left(\frac{-9}{4} - \frac{27}{4}\right) - j(0 - 0) + k(0 - 0)$$

$$= \langle -9, 0, 0\rangle$$

$\kappa(t) = \dfrac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \dfrac{1}{81}$

$|r'(t) \times r''(t)| = 9$

$|r'(t)|^3 = \sqrt{0^2 + (\frac{3}{2})^2 + (-\frac{3\sqrt{3}}{2})^2} = 9^3$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans.

---

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$grad(f) = \langle y^2z^3, \ 2xyz^3, \ 3xy^2z^2 \rangle = \langle 1, 2, 3 \rangle$

$grad(g) = \langle 1, \ 2y, \ 3z^2 \rangle = \langle 1, 2, 3 \rangle$

$grad(f) \cdot grad(g)$

$= \langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle$

$= 1 + 4 + 9 = \boxed{14}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} .$$

---

ans. 0

$$x^2-y^2 = (x+y)(x-y)$$

$$(x+y)^2-(z+w)^2 = (x+y+z+w)(x+y-z-w)$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y+z+w)\cancel{(x+y-z-w)}}{\cancel{x+y-z-w}} = 0$$