NAME: (print!) SAI EMBAR **RUID: (print!)** 193003278

SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$
3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$
4. $\langle 3,-6,9\rangle$
5. $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$
6. $-4\sqrt{3}$
7. $6\cos^2(1) - 6\sin^2(1)$
8. 8π
9. -15
10. $(\frac{3}{4}, -1)$ is a saddle point
11. 3.00033
12. $\frac{\sqrt{2}}{6}$
13. $\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} \rho^6 \sin^4\phi \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\theta\, d\phi$
14. $\frac{1}{3}$
15. $\int_0^1\int_0^v 2\sqrt{v^4+4v^2u^2+u^4}\, du\, dv$
16. 14
17. 0

Sign the following declaration:

I SAI EMBAR Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: E Sai

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the vector-field line integral

$$\int_C (\cos(e^{\sin x}) + 5y)\, dx + (\sin(e^{\cos y}) + 11x)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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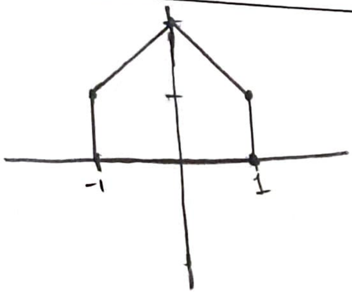
ans. -18



Clockwise direction ↻

Green's Theorem

$\int_C P(x,y)dx + Q(x,y)dy$ $\qquad$ $P = \cos(e^{\sin x}) + 5y$, $Q = \sin(e^{\cos y}) + 11x$

$=$

$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$\frac{\partial Q}{\partial x} = 11$, $\frac{\partial P}{\partial y} = 5$

$\iint_D (11-5)\,dA = 6\iint_D dA$

$= 6 \cdot \text{Area}(D)$

Area(D) = Area of Rectangle + Area of triangle above rectangle

$=$ base (height) $+ \frac{1}{2}$ (base)(height)

$= 2(1) + \frac{1}{2}(2)(1) \to \text{Area}(D) = 2 + 1 = 3$

So, $6 \cdot \text{Area}(D) = 6 \cdot 3 = 18 \cdot -1$ ← clockwise $= \boxed{-18}$

2. (12 points) Change the order of integration

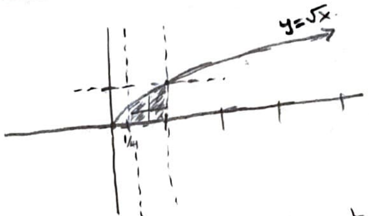
$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx \quad .$$

ans. $\int_0^{1/2} \int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

$x = \frac{1}{4}$ $\quad y = 0$

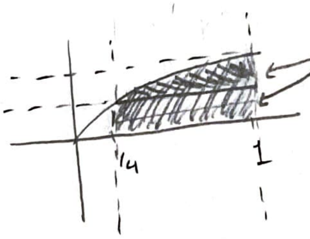
$x = 1$ $\quad y = \sqrt{x}$

$x = y^2$



Change to type II (horizontally simple)

However, since the region does not start at $x = 0$ and instead starts at $x = \frac{1}{4}$, we cannot change the order of integration with 1 double integral.



Split area into two sections first is the rectangle, then the area above it. By finding the intersection between $x = \frac{1}{4}$ and $x = y^2$, we can write the double integral far easier by splitting it up.

To find intersection point between $x = 1/4$ and $x = y^2$, set them equal and solve.

$\frac{1}{4} = y^2 \to y = \frac{1}{2}$

For area of rectangle:

y bounds go from 0 to $\frac{1}{2}$

x bounds go from $\frac{1}{4}$ to 1

so, $\int_0^{1/2} \int_{1/4}^{1} f(x,y)\,dx\,dy$

For area above rectangle:

y bounds go from $1/2$ to 1

x bounds go from y^2 to 1.

so, $\int_{1/2}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

Now, add them together.

$$\boxed{\int_0^{1/2} \int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy}$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x = -2\sin(x+y) - 4\sin(x+z)$

$f_y = -2\sin(x+y) - 8\sin(y+z)$

$f_z = -4\sin(x+z) - 8\sin(y+z)$

plug in point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$

$f_x = -2\sin(\frac{\pi}{6}+\frac{\pi}{6}) - 4\sin(\frac{\pi}{6}+\frac{\pi}{6})$

$= -2\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3})$

$= -2(\frac{\sqrt{3}}{2}) - 4(\frac{\sqrt{3}}{2}) = -\sqrt{3} - 2\sqrt{3} = \underline{-3\sqrt{3}}$

$f_y = -2\sin(\frac{\pi}{6}+\frac{\pi}{6}) - 8\sin(\frac{\pi}{6}+\frac{\pi}{6})$

$= -2\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})$

$= -2(\frac{\sqrt{3}}{2}) - 8(\frac{\sqrt{3}}{2}) = -\sqrt{3} - 4\sqrt{3} = \underline{-5\sqrt{3}}$

$f_z = -4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})$

$= -4(\frac{\sqrt{3}}{2}) - 8(\frac{\sqrt{3}}{2}) = -2\sqrt{3} - 4\sqrt{3} = \underline{-6\sqrt{3}}$

<u>Now write equation of tangent plane</u>

$-3\sqrt{3}(x-\frac{\pi}{6}) - 5\sqrt{3}(y-\frac{\pi}{6}) - 6\sqrt{3}(z-\frac{\pi}{6}) = 0$

divide both sides by $\sqrt{3}$

$-3(x-\frac{\pi}{6}) - 5(y-\frac{\pi}{6}) - 6(z-\frac{\pi}{6}) = 0$

$-3x + \frac{3\pi}{6} - 5y + \frac{5\pi}{6} - 6z + \frac{6\pi}{6} = 0$

$-3x - 5y - 6z + \frac{14\pi}{6} = 0$

$6z = -3x - 5y + \frac{14\pi}{6}$

divide by 6

$z = -\frac{1}{2}x - \frac{5}{6}y + \frac{14\pi}{36}$

$\boxed{z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}}$

4. (16 points) Let **a**, **b**, **c** be three vectors such that

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k}\ ,\quad \mathbf{b}\times\mathbf{c} = \mathbf{i}-\mathbf{j}+\mathbf{k}\ ,\quad \mathbf{a}\times\mathbf{c} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}\ .$$

What is

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\times(2\mathbf{a}-\mathbf{b}+3\mathbf{c}) \quad ?$$

---

ans. $\langle 3,-6,9\rangle$

We can use the property $a\times(b+c)=(a\times b)+(a\times c)$

$a\times(2a-b+3c)+b\times(2a-b+3c)+c\times(2a-b+3c)$

$\underbrace{(a\times 2a)}_{0}-(a\times b)+(a\times 3c)+(b\times 2a)-\underbrace{(b\times b)}_{0}+(b\times 3c)+(c\times 2a)-(c\times b)+\underbrace{(c\times 3c)}_{0}$

$-(a\times b)+(a\times 3c)+(b\times 2a)+(b\times 3c)+(c\times 2a)-(c\times b)$

We can use property $a\times rc = r(a\times c)$ and $a\times b = -(b\times a)$

$-\langle 1,1,-1\rangle+3\langle 2,1,2\rangle-2\langle 1,1,-1\rangle+3\langle 1,-1,1\rangle-2\langle 2,1,2\rangle+\langle 1,-1,1\rangle$

$\langle -1,-1,1\rangle+\langle 6,3,6\rangle+\langle -2,-2,2\rangle+\langle 3,-3,3\rangle+\langle -4,-2,-4\rangle+\langle 1,-1,1\rangle$

all x components added $= -1+6-2+3-4+1 = 3$

all y components added $= -1+3-2-3-2-1 = -6$

all z components added $= 1+6+2+3-4+1 = 9$

$= \boxed{\langle 3,-6,9\rangle}$

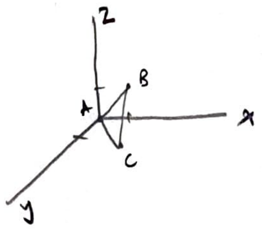
5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;



We can use the formula $\vec{v} \cdot \vec{w} = \|\vec{v}\| \|\vec{w}\| \cos\theta$ and manipulate it to get

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|}$$

$\vec{AB} = \langle 1, 0, 1 \rangle$

$\vec{BC} = \langle 0, 1, -1 \rangle$

$\vec{AC} = \langle 1, 1, 0 \rangle$

$\vec{BA} = \langle -1, 0, -1 \rangle$

angle at A: $\cos\theta = \frac{\vec{AB} \cdot \vec{AC}}{\|\vec{AB}\| \|\vec{AC}\|}$

$\cos\theta = \frac{\langle 1,0,1 \rangle \cdot \langle 1,1,0 \rangle}{\sqrt{2} \cdot \sqrt{2}}$

$\cos\theta = \frac{1+0+0}{2} = \frac{1}{2}$

$\theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ rad

angle at B: $\cos\theta = \frac{\vec{BA} \cdot \vec{BC}}{\|\vec{BA}\| \|\vec{BC}\|}$

$\cos\theta = \frac{\langle -1, 0, -1 \rangle \cdot \langle 0, 1, -1 \rangle}{\sqrt{2} \cdot \sqrt{2}}$

$\cos\theta = \frac{0+0+1}{2} = \frac{1}{2}$

$\theta = \cos^{-1}(\frac{1}{2}) = \frac{\pi}{3}$ rad

sum of angles in a triangle is 180° which is π radians, so, to find angle at C we subtract $\pi - \frac{\pi}{3} - \frac{\pi}{3}$

= angle at C: $\frac{\pi}{3}$ rad

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

ans. $-4\sqrt{3}$

$f_x = 3x^2 + yz$

$f_y = 3y^2 + xz$

$f_z = 3z^2 + xy$

plug in $P = (1,1,1)$

$f_x = 3+1=4$

$f_y = 3+1=4$

$f_z = 3+1=4$

$\nabla f = \langle 4,4,4 \rangle$

$P = (1,1,1)$

$Q = (-1,-1,-1)$

$\vec{PQ} = \langle -2,-2,-2 \rangle$

$\|\vec{PQ}\| = \sqrt{(-2)^2+(-2)^2+(-2)^2} = 2\sqrt{3}$

$u_{PQ} = \left\langle \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}} \right\rangle$

$D_{\vec{u}} f(x,y,z) = \vec{\nabla} f(x,y,z) \cdot \vec{u}$

$= \langle 4,4,4 \rangle \cdot \left\langle \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}}, \frac{-2}{2\sqrt{3}} \right\rangle$

$= \frac{-8}{2\sqrt{3}} - \frac{8}{2\sqrt{3}} - \frac{8}{2\sqrt{3}}$

$= \frac{-4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}}$

$= \frac{-12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-\cancel{12}^{\,4}\sqrt{3}}{\cancel{3}} = \boxed{-4\sqrt{3}}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos^2(1) - 6\sin^2(1)$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial g}{\partial y}\left(\frac{\partial y}{\partial u}\right)$$

$$\frac{\partial x}{\partial u} = e^u \cos v$$

$$\frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial g}{\partial x} = 6x$$

$$\frac{\partial g}{\partial y} = -6y$$

$$\frac{\partial g}{\partial u} = 6x(e^u \cos v) + (-6y)(e^u \sin v)$$

$$= 6(e^u \cos v)(e^u \cos v) + (-6(e^u \sin v)(e^u \sin v)$$

at $(0,1)$

$$= 6(e^0 \cos(1))(e^0 \cos(1)) - 6(e^0 \sin(1))(e^0 \sin(1))$$

$$= 6\cos(1)\cos(1) - 6\sin(1)\sin(1)$$

$$= \boxed{6\cos^2(1) - 6\sin^2(1)}$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + \cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \ \ and \ \ x > 0 \ \ and \ \ y < 0 \ \ and \ \ z > 0\} \ .$$

---

ans. 8π

Divergence Theorem

$P = 3x + \cos(y^3 + yz)$

$Q = -2y + e^{x+z^2}$

$R = 5z + \sin(xy^3 + e^x)$

$\operatorname{div} F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

$= \frac{\partial}{\partial x}(3x + \cos(y^3 + yz)) + \frac{\partial}{\partial y}(-2y + e^{x+z^2}) + \frac{\partial}{\partial z}(5z + \sin(xy^3 + e^x))$

$\operatorname{div} F = 3 + (-2) + 5$

$\operatorname{div} F = 6$

The region is a 1/8 of a sphere. The function $x^2+y^2+z^2<4$ describes a sphere, but $x>0$, $y<0$, $z>0$ bounds it to one octant, so it is 1/8 of a sphere. The radius is 2 based on the equation $x^2+y^2+z^2<4$.

Divergence Theorem

$\iiint_E 6 \, dV = 6 \cdot \text{Vol}(E)$

Integrand is constant so use formula: $\frac{4}{3}\pi r^3$

$r=2$, so

$\frac{4}{3}\pi(2)^3 = \frac{32\pi}{3}(6) = 64\pi$

divide by 8 to find volume of 1/8 sphere.

$\frac{64\pi}{8} = \boxed{8\pi}$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z \, , \, 2x \, , \, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

ans. -15

$P = 3z$, $Q = 2x$, $R = y+z$

$g(x,y) = z$

Stokes Theorem

$\frac{\partial g}{\partial x} = 2$, $\frac{\partial g}{\partial y} = 3$

$$\iint_S F \cdot ds = \iint_D \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dA$$

$$\iint_D \left(-3z(2) - 2x(3) + y + z\right) dA$$

bounds are given, also replace z with $g(x,y)$

$$\int_0^1 \int_0^1 (-3(2x+3y)(2) - 2x(3) + y + 2x + 3y) \, dx \, dy$$

$$= \int_0^1 \int_0^1 (-6(2x+3y) - 6x + y + 2x + 3y) \, dx \, dy$$

$$= \int_0^1 \int_0^1 (-12x - 18y - 6x + y + 2x + 3y) \, dx \, dy = \int_0^1 \int_0^1 (-16x - 14y) \, dx \, dy$$

$$= \int_0^1 (-16x - 14y) dx = -\frac{16x^2}{2} - 14xy \Big|_0^1 = -8 - 14y$$

$$= \int_0^1 (-8 - 14y) dy = \left(-8y - \frac{14y^2}{2}\right)\Big|_0^1 = -8 - 7 = \boxed{-15}$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. $\left(\frac{3}{4}, -1\right)$ is a saddle point

$f_x = 4 - \frac{1}{2x+y}(2)$

$f_y = -2y - \frac{1}{2x+y}$

set $f_x = 0$: $4 - \frac{2}{2x+y} = 0$

$-\frac{2}{2x+y} = -4$

$\frac{2}{2x+y} = 4$

$2 = 4(2x+y)$

$\frac{1}{2} = 2x+y$

set $f_y = 0$: $-2y - \frac{1}{2x+y} = 0$

$-2y - \frac{1}{1/2} = 0$

$-2y - 2 = 0$

$-2y = 2$

$y = -1$

plug back in to find x

$\frac{1}{2} = 2x - 1$

$\frac{3}{2} = 2x$

$x = \frac{3}{4}$

CP: $\left(\frac{3}{4}, -1\right)$

$f_{xx} = \frac{4}{(2x+y)^2}$

$f_{xy} = \frac{2}{(2x+y)^2}$

$f_{yy} = -2 + \frac{1}{(2x+y)^2}$

@ $\left(\frac{3}{4}, -1\right)$

$f_{xx} = \frac{4}{(\frac{3}{2}-1)^2} = 16$

$f_{xy} = \frac{2}{(\frac{3}{2}-1)^2} = 8$

$f_{yy} = -2 + \frac{1}{(\frac{3}{2}-1)^2} = 2$

$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$

$= 16(2) - (8)^2$

$= 32 - 64 = -32$

$\boxed{\text{Determinant is } (-) \text{ negative so } \left(\tfrac{3}{4}, -1\right) \text{ is a saddle point}}$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad .$$

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ans. 3.00033

$$f_x = \frac{\partial}{\partial x}(2x^2+3y^2+z^2)^{1/2} = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_y = \frac{\partial}{\partial y}(2x^2+3y^2+z^2)^{1/2} = \frac{3y}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_z = \frac{\partial}{\partial z}(2x^2+3y^2+z^2)^{1/2} = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

$$f(1,1,2) = \sqrt{2(1)^2+3(1)^2+2^2} = \sqrt{9} = 3$$

$$f_x(1,1,2) = \frac{2(1)}{\sqrt{9}} = \frac{2}{3}$$

$$f_y(1,1,2) = \frac{3(1)}{\sqrt{9}} = 1$$

$$f_z(1,1,2) = \frac{2}{\sqrt{9}} = \frac{2}{3}$$

$$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$$

plug in $f(1.001, 0.999, 2.001)$

$$f(1.001, 0.999, 2.001) \approx 3 + \frac{2}{3}(1.001-1) + 1(0.999-1) + \frac{2}{3}(2.001-2)$$

$$f(1.001, 0.999, 2.001) \approx \boxed{3.000333}$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

ans. $\frac{\sqrt{2}}{6}$

first integral

$y=0$, $y=x$

$r\sin\theta=0$ | $r\sin\theta=r\cos\theta$

$\sin\theta=0$ | $1=\tan\theta$

$\theta=0$ | $\theta=\frac{\pi}{4}$

replace x and y into polar, solve for θ to get θ bounds, replace other bounds with polar and solve for r to get r bounds.

$x=0$, $x=\frac{\sqrt{2}}{2}$

$r\cos\theta=\frac{\sqrt{2}}{2}$

$r=\frac{\sqrt{2}}{2\cos\theta}$

$$\int_0^{\sqrt{2}/2}\int_0^x x\,dy\,dx=\int_0^{\pi/4}\int_0^{\frac{\sqrt{2}}{2\cos\theta}} r^2\cos\theta\,dr\,d\theta$$

$$=\int_0^{\frac{\sqrt{2}}{2\cos\theta}} r^2\cos\theta\,dr=\frac{r^3}{3}\cos\theta\,dr\Big|_0^{\frac{\sqrt{2}}{2\cos\theta}}$$

$$=\frac{\left(\frac{\sqrt{2}}{2\cos\theta}\right)^3}{3}\cos\theta=\frac{\frac{2\sqrt{2}}{8\cos^3\theta}}{3}\cos\theta=\frac{2\sqrt{2}}{24\cos^3\theta}\cos\theta$$

$$=\frac{\sqrt{2}}{12\cos^2\theta}$$

$$=\int_0^{\pi/4}\frac{\sqrt{2}}{12\cos^2\theta}\,d\theta=\frac{\sqrt{2}}{12}\int_0^{\pi/4}\frac{1}{\cos^2\theta}\,d\theta$$

standard integral of $\sec^2(\theta)=\tan(\theta)$

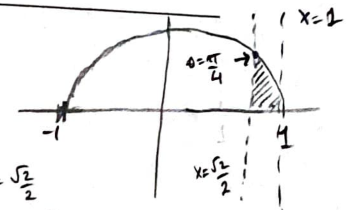
$$=\frac{\sqrt{2}}{12}(\tan(\theta))\Big|_0^{\pi/4}$$

$$=\frac{\sqrt{2}}{12}$$

$y=0$, $y=\sqrt{1-x^2}$

$y^2+x^2=1$

so $r=1$



$x=\frac{\sqrt{2}}{2}$, $x=1$

$r\cos\theta=\frac{\sqrt{2}}{2}$ | $r\cos\theta=1$

$\theta=\frac{\pi}{4}$ | $\theta=0$ ← theta bounds.

$x=\frac{\sqrt{2}}{2}$

$r\cos\theta=\frac{\sqrt{2}}{2}$

$r=\frac{\sqrt{2}}{2\cos\theta}$ ← lower bound because it doesn't start at 0.

$$\int_{\sqrt{2}/2}^{1}\int_0^{\sqrt{1-x^2}} x\,dy\,dx=\int_0^{\pi/4}\int_{\frac{\sqrt{2}}{2\cos\theta}}^{1} r^2\cos\theta\,dr\,d\theta$$

$$=\int_{\frac{\sqrt{2}}{2\cos\theta}}^{1} r^2\cos\theta\,dr=\left(\frac{r^3}{3}\cos\theta\right)\Big|_{\frac{\sqrt{2}}{2\cos\theta}}^{1}$$

$$=\frac{\cos\theta}{3}-\frac{\sqrt{2}}{12\cos^2\theta}$$

$$=\int_0^{\pi/4}\frac{\cos\theta}{3}-\frac{\sqrt{2}}{12\cos^2\theta}\,d\theta$$

$$=\left(\frac{\sin\theta}{3}-\frac{\sqrt{2}}{12}(\tan\theta)\right)\Big|_0^{\pi/4}$$

$$=\frac{\sqrt{2}}{6}-\frac{\sqrt{2}}{12}=\frac{2\sqrt{2}}{12}-\frac{\sqrt{2}}{12}=\frac{\sqrt{2}}{12}$$

Add the two integrals together.

$$\frac{\sqrt{2}}{12}+\frac{\sqrt{2}}{12}=\frac{2\sqrt{2}}{12}=\boxed{\frac{\sqrt{2}}{6}}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to spherical coordinates. Do not evaluate.

---

ans. $\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \rho^6 \sin^4\phi \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\theta\, d\phi$

$0 \le z \le 2$ → describes right half of top half of circle with radius 2.

$0 \le y \le \sqrt{4-z^2}$ → top half of circle with radius 2.

$-\sqrt{4-z^2-y^2} \le x \le 0$

R^2

$\rho = R$

$\rho = 2$

$0 \le \theta \le \frac{\pi}{2}$ → because the object is constrained to top right quadrant as z is (+).

$2\cos\varphi = 0$ → plug in 0 because $x=0$ in the bounds.

$\cos\varphi = 0$

$\varphi = \frac{\pi}{2}$

$0 \le \phi \le \frac{\pi}{2}$

$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$
$dV = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} (\rho\sin\phi\cos\theta)^2 (\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \rho^2\sin^2\phi\cos^2\theta(\rho\sin\phi\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\, d\rho\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{2} \rho^6\sin^4\phi\cos^2\theta\sin\theta\cos\phi\, d\rho\, d\theta\, d\phi$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$

$r'(t) \times r''(t) = \langle 0, 3\cos t, -3\sin t\rangle \times \langle 0, -3\sin t, -3\cos t\rangle$

$$= \begin{vmatrix} i & j & k \\ 0 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix} = (-9\cos^2 t - 9\sin^2 t)\hat{i} - 0\hat{j} + 0\hat{k}$$

$= -9(\cos^2 t + \sin^2 t)\hat{i}$

$= -9(1)\hat{i}$

$= -9$

$\|r'(t) \times r''(t)\| = \sqrt{(-9)^2} = 9$

$\|r'(t)\| = \sqrt{0^2 + (3\cos t)^2 + (-3\sin t)^2}$

$= \sqrt{9\cos^2 t + 9\sin^2 t}$

$= \sqrt{9(1)} = 3$

$$K = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$$= \frac{9}{3^3} = \frac{9}{27} = \boxed{\frac{1}{3}}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \ .$$

---

ans. $\int_0^1 \int_0^v 2\sqrt{v^4 + 4v^2u^2 + u^4}\, du\, dv$

$dS = \|\vec{N}\|\, du\, dv$

$\vec{T}_u = \frac{\partial G}{\partial u} = \langle 2u, v, 0\rangle$

$\vec{T}_v = \frac{\partial G}{\partial v} = \langle 0, u, 2v\rangle$

$\vec{N} = \vec{T}_u \times \vec{T}_v = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = (2v^2)\hat{i} - (4uv)\hat{j} + (2u^2)\hat{k}$

$\|\vec{N}\| = \sqrt{(2v^2)^2 + (-4uv)^2 + (2u^2)^2} = \sqrt{4v^4 + 16u^2v^2 + 4u^4} = 2\sqrt{v^4 + 4v^2u^2 + u^4}$

$$\boxed{\int_0^1 \int_0^v 2\sqrt{v^4 + 4v^2u^2 + u^4}\, du\, dv}$$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) . grad(g) \quad .$$

at the point (1, 1, 1).

ans. 14

$f_x = y^2z^3$
$f_y = 2xyz^3$
$f_z = 3xy^2z^2$
$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$
at $(1,1,1)$
$\nabla f = \langle 1, 2, 3 \rangle$

$g_x = 1$
$g_y = 2y$
$g_z = 3z^2$
$\nabla g = \langle 1, 2y, 3z^2 \rangle$
at $(1,1,1)$
$\nabla g = \langle 1, 2, 3 \rangle$

$\nabla f \cdot \nabla g = \langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle$
$= 1 + 4 + 9 = \boxed{14}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. 0

first plug in (0,0,0,0)

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \left[\frac{0}{0}\right]$$

Cannot simplify

check limit at different paths approaching function.

$(0,0,0,w) \to$ w-axis line.

$$\lim_{w\to 0} \frac{(0+0)^2-(0+w)^2}{0+0-0-w}$$

$$\lim_{w\to 0} \frac{-w^2}{-w} = \lim_{w\to 0} w = 0$$

Check at $(0,0,z,0) \to$ z-axis line

$$\lim_{z\to 0} \frac{0^2-(z+0)^2}{0+0-z-0} = \lim_{z\to 0} \frac{-z^2}{-z} = \lim_{z\to 0} z = 0$$

Check at $(0,y,0,0) \to$ y-axis line

$$\lim_{y\to 0} \frac{y^2-0^2}{0+y-0-0} = \lim_{y\to 0} \frac{y^2}{y} = \lim_{y\to 0} y = 0$$

check at $(x,0,0,0) \to$ x-axis line

$$\lim_{x\to 0} \frac{x^2-0^2}{x+0-0-0} = \lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

Approaching (0,0,0,0) the limit is $\boxed{0}$ because you get the same limit when approaching from different lines on different axes.