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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

1.

2. $\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

3. $-\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

4. $\langle 3, -6, 9 \rangle$

5. $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$

6. $-4\sqrt{3}$

7. $g_v(0,1) = 6\cos(2)$

8.

9.

10.

11. $9001/3000$

12.

13.

14.

15.

16.

17.

Sign the following declaration:

I   Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x,y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \,+\, (sin\,(e^{\cos y})\,+\,11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

Greene's Theorem

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

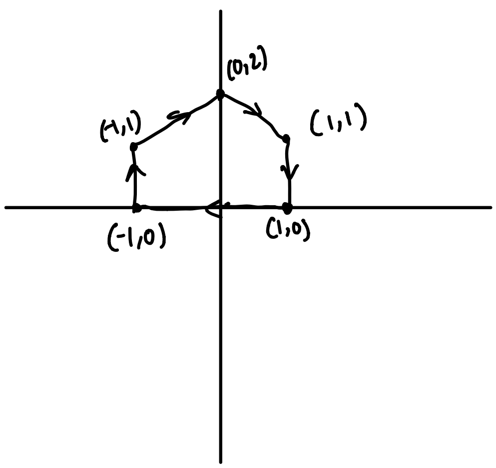
clockwise

mult (-1)

Explain!

---

ans. $6\sqrt{25+10\sqrt{5}}$



$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$P = \cos(e^{\sin x}) + 5y$

$P_y = 5$

$Q = \sin(e^{\cos y}) + 11x$

$Q_x = 11$

$$\iint_D (11-5)\,dx\,dy$$

$$\iint \underline{6}\,dx\,dy$$

$6 \times$ Area D

$6\left(\frac{1}{4}\sqrt{5(5+2\sqrt{5})}\;a^2\right)$

$6\left(\sqrt{5(5+2\sqrt{5})}\right)$

$= 6\sqrt{25+10\sqrt{5}}$

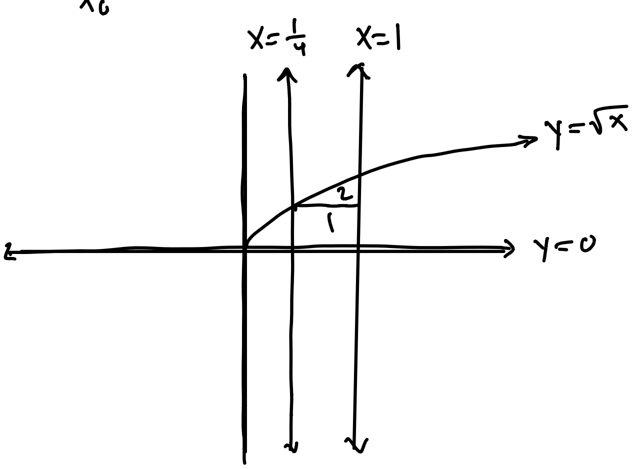
2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, \overset{dy\ dx}{\underline{dx}\, dy} \qquad .$$

---

ans. $\displaystyle \int_{0}^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

$$\int_{x_0}^{x} \int_{y_0}^{y} f(x,y)\,dx\,dy$$



Split into two integrals

1) $\displaystyle \int_{0}^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy$

2) $\displaystyle \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

Add both areas

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \; = \; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z)$

$f_x = -2(2\sin(x+z) + \sin(x+y))$

$f_y = -2(4\sin(y+z) + \sin(x+y))$

$f_z = -4(2\sin(y+z) + \sin(x+z))$

$f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -3\sqrt{3}$

$f_y(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -5\sqrt{3}$

$f_z(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -6\sqrt{3}$

$-3\sqrt{3}(x-\frac{\pi}{6}) - 5\sqrt{3}(y-\frac{\pi}{6}) - 6\sqrt{3}(z-\frac{\pi}{6}) = 0$

$-3(x-\frac{\pi}{6}) - 5(y-\frac{\pi}{6}) - 6(z-\frac{\pi}{6}) = 0$

$-3x - 5y - 6z + \frac{7\pi}{3} = 0$

$6z = -3x - 5y + \frac{7\pi}{3}$

$z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\underbrace{\mathbf{a}\times\mathbf{b}}_{b \times a} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b}\times\mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a}\times\mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a}+\mathbf{b}+\mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $\langle 3, -6, 9 \rangle$

$$a \times (2a - b + 3c) + b \times (2a - b + 3c) + c \times (2a - b + 3c)$$

$$\cancel{a \times 2a}^{0} - a \times b + a \times 3c$$

$$+ b \times 2a - \cancel{b \times b}^{0} + b \times 3c$$

$$+ c \times 2a - c \times b + \cancel{c \times 3c}^{0}$$

$$- a \times b + a \times 3c + b \times 2a + b \times 3c + c \times 2a - c \times b$$

$$\underbrace{\langle -1,-1,1 \rangle + \langle 6,3,6 \rangle} + \underbrace{\langle -2,-2,2 \rangle + \langle 3,-3,3 \rangle} + \langle -4,-2,-4 \rangle$$

$$+ \langle 1,-1,1 \rangle$$

$$\langle 5, 2, 7 \rangle + \langle 1, -5, 5 \rangle + \langle -3, -3, -3 \rangle$$

$$\langle 6, -3, 12 \rangle + \langle -3, -3, -3 \rangle$$

$$\langle 3, -6, 9 \rangle$$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;

$\overline{AB} = (1,0,1) - (0,0,0) = \langle 1,0,1 \rangle$, $|\overline{AB}| = \sqrt{2}$

$\overline{AC} = (1,1,0) - (0,0,0) = \langle 1,1,0 \rangle$, $|\overline{AC}| = \sqrt{2}$

$\overline{BC} = (1,1,0) - (1,0,1) = \langle 0,1,-1 \rangle$, $|\overline{BC}| = \sqrt{2}$

$\overline{BA} = (0,0,0) - (1,0,1) = \langle -1,0,-1 \rangle$

$$\cos\theta_A = \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}| \cdot |\overline{AC}|} = \frac{(1+0+0)}{\sqrt{2} \cdot \sqrt{2}}$$

$$\cos\theta_A = \frac{1}{2}$$

$$\theta_A = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\cos\theta_B = \frac{\overline{BC} \cdot \overline{BA}}{|\overline{BC}| \cdot |\overline{BC}|} = \frac{(0+0+1)}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

$$\theta_B = \cos^{-1}\left(\frac{1}{2}\right)$$

$$\theta_C = \pi - \theta_A - \theta_B$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

P ↓ Q

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans. $-4\sqrt{3}$

$$\nabla f(x,y,z) = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle$$

$$\overline{PQ} = Q - P = (-1,-1,-1) - (1,1,1) = \langle -2,-2,-2 \rangle$$

$$|\langle -2,-2,-2 \rangle| = \sqrt{4+4+4} = \sqrt{12} = 2\sqrt{3}$$

$$u = \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$$

$$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$$

$$\nabla f(1,1,1) \cdot u$$

$$\langle 4,4,4 \rangle \cdot \langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$$

$$= -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}}$$

$$= -\frac{12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{12\sqrt{3}}{3}$$

$$= -4\sqrt{3}$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $g_u(0,1) = 6\cos(2)$

$g_u = (g_x)(x_u) + (g_y)(y_u)$

$g_x = 6x \,,\ g_y = -6y \rightarrow g_x = 6e^u \cos v \,,\ g_y = -6e^u \sin v$

$x_u = e^u$

$g_u = (6e^u \cos v)(e^u \cos v) - (6e^u \sin v)(e^u \sin v)$

$g_u(0,1) = 6\cos^2(1) - 6\sin^2(1)$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

$x^2+y^2+z^2 < 4$

$0 \leq \rho \leq 2$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \phi \leq \frac{\pi}{2}$

$\iint \mathbf{F}\cdot d\mathbf{S} = \iiint_D \operatorname{div} \mathbf{F}\, dV$

$\operatorname{div} \mathbf{F} = 3 - 2 + 5$

$\iiint_D 6 \quad dV$

$6\left(\frac{4}{3}\pi r^3\right) \leftarrow$ Volume of a sphere

$6\left(\frac{4}{3}\pi(4)\right)$

$\frac{32\pi}{8}$ (

$= 4\pi$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 \le x \le 1, \quad 0 \le y \le 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15

$$\iint_S F.dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$F = \langle \underset{P}{3z}, \underset{Q}{2x}, \underset{R}{y+z} \rangle$$

$$g(x,y) = 2x + 3y \leftarrow$$

$$g_x = 2, \quad g_y = 3$$

$$\int_0^1 \int_0^1 \left(-6z - 6x + y + z\right) dx\, dy$$

$$\int_0^1 \int_0^1 \left(-12x - 6y - 6x + y + 2x + 3y\right) dx\, dy$$

$$\int_0^1 \int_0^1 \left(-16x - 14y\right) dx\, dy = -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. Critical Point: $(\frac{3}{4}, -1)$, Saddle point: $(\frac{3}{4}, -1)$

$f_x = 4 - \frac{2}{2x+y}$, $f_y = -2y - \frac{1}{2x+y}$

$4 - \frac{2}{2x+y} = 0$, $-2y - \frac{1}{2x+y} = 0$

Critical Points @ $(\frac{3}{4}, -1)$

$f_{xx} = \frac{4}{(2x+y)^2}$, $f_{yy} = \frac{1}{(y+2x)^2} - 2$

$f_{xy} = \frac{2}{(y+2x)^2}$

Find the determinant, Use second derivative test.

$D = f_{xx} f_{yy} - [f_{xy}]^2$

$(\frac{3}{4}, -1)$

$f_{xx} = 16$, $f_{yy} = 2$, $f_{xy} = 8$

$D = (16)(2) - 64$

$= -32 < 0$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. $\frac{9001}{3000}$

$(a, b, c) = (1, 1, 2)$

$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} \Rightarrow f_x(1,1,2) = \frac{2}{3}$

$f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} \Rightarrow f_y(1,1,2) = 1$

$f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}} \Rightarrow f_z(1,1,2) = \frac{2}{3}$

$L(x,y,z) = f(a,b,c) + f_x(x-a) + f_y(y-b) + f_z(z-c)$

$L(x,y,z) = 3 + \frac{2}{3}(x-1) + (y-1) + \frac{2}{3}(z-2)$

$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(1.001-1) + (.999-1) + \frac{2}{3}(2.001-2)$

$L(1.001, 0.999, 2.001) \approx \frac{9001}{3000}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\underbrace{\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx}_{1} + \underbrace{\int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx}_{2} \quad .$$

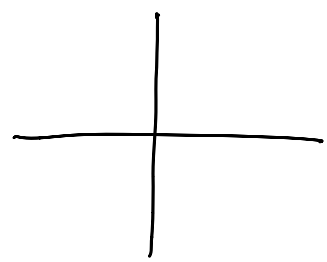
Explain!

---

ans. $\frac{1}{3\sqrt{2}}$

$x^2 + y^2 = 1$

$x = \cos\theta, \; y = \sin\theta$



1) $\int_0^{\frac{\pi}{4}} \int_0^{\cos\theta} \cos\theta \cdot r\,dr\,d\theta$

2) $\int^{\frac{\pi}{4}} \int_0^{\pi} \cos\theta \cdot r\,dr\,d\theta$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\sqrt{2}}{2\cos\theta}} r\cos\theta \cdot r\,dr\,d\theta + \int_0^{\frac{\pi}{4}} \int_0^{\frac{\sqrt{2}}{2\cos\theta}} r\cos\theta \cdot r\,dr\,d\theta$$

13. (12 points) Convert the triple iterated integral

$$d\phi \, d\theta \, d\rho$$

$$\int_0^{\cancel{4}\;2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 \, y \, z \, dx \, dy \, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\displaystyle\int_0^{\pi/2} \int_{\frac{\pi}{2}}^{\pi} \int_0^{2} (\rho^2 \sin^2\theta \cos^2\theta)(\rho \sin^2\theta)(\rho \cos\phi)\, \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$

$$x^2 + y^2 + z^2 = 4$$

$$0 \le \rho \le 2$$

$$\frac{\pi}{2} \le \theta \le \pi$$

$$0 \le \phi \le \frac{\pi}{2}$$

$$\int_0^{\pi/2} \int_{\frac{\pi}{2}}^{\pi} \int_0^{2} (\rho^2 \sin^2\theta \cos^2\theta)(\rho \sin^2\theta)(\rho \cos\phi)\, \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{1}{3}$

$$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$$

$$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$$

$$r'\left(\frac{\pi}{3}\right) = \left\langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2} \right\rangle$$

$$r''\left(\frac{\pi}{3}\right) = \left\langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2} \right\rangle$$

$$r'\left(\frac{\pi}{3}\right) \times r''\left(\frac{\pi}{3}\right) = \langle -9, 0, 0 \rangle$$

$$|\langle -9, 0, 0 \rangle| = 9$$

$$\left|r'\left(\frac{\pi}{3}\right)\right| = \sqrt{0 + \frac{9}{4} + \frac{27}{4}} = 3$$

$$\kappa\left(\frac{\pi}{3}\right) = \frac{\left|r'\left(\frac{\pi}{3}\right) \times r''\left(\frac{\pi}{3}\right)\right|}{\left|r'\left(\frac{\pi}{3}\right)\right|^3}$$

$$= \frac{9}{27} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parametrically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans.

$$\int \int f(x,y) \, dy \, dx$$

$x = u^2$, $y = uv$, $z = v^2$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$\nabla f = \langle y^2z^3, \ 2xyz^3, \ 3xy^2z^2 \rangle$

$\nabla f(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, \ 2y, \ 3z^2 \rangle$

$\nabla g(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla f(1,1,1) \cdot \nabla g(1,1,1)$

$\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle$

$1 + 4 + 9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. 0

Plug in $(0,0,0,0)$, we get $\frac{0}{0}$

Simplify with algebra

$$\lim_{f(x,y,z,w)\to(0,0,0,0)} \frac{\cancel{(x+y-z-w)}(x+y+z+w)}{\cancel{x+y-z-w}}$$

Cancels out due to diff of squares

$$\lim_{f(x,y,z,w)\to(0,0,0,0)} (x+y+z+w) = 0$$

Limit exists!