8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \ .$$

---

ans. 512π

Since closed surface, use Div. Thm:

$$\iint_S F \cdot dS = \iiint_E \operatorname{div}(F)\, dV$$

$$\operatorname{div}(F) = 3 + (-2) + 5 = 6$$

$$\text{Ans} = \iiint_E 6\, dV$$

E: Sphere w/ $0 \le \rho \le 4$, $\pi \le \theta \le 2\pi$, $0 \le \phi \le \frac{\pi}{2}$

$$\text{Vol} = \frac{4}{3}\pi r^3 = \frac{256\pi}{3}$$

$$= \text{Vol} \cdot 6 = \boxed{512\pi}$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15

$$= \iint_D \left(-P \frac{\partial z}{\partial x} - Q \frac{\partial z}{\partial y} + R\right) dA$$

$$= \int_0^1 \int_0^1 \left(-3(2x+3y)\cdot 2 - (2x)(3) + y + (2x+3y)\right) dx\,dy$$

$$= \int_0^1 \int_0^1 -12x - 6x + 2x - 18y + y + 3y \; dx\,dy$$

$$= \int_0^1 \int_0^1 -16x - 14y \, dx\,dy$$

$$= -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. $\left(\frac{1}{4}, 0\right)$ is a saddle point

$$\nabla f = \left\langle 4 - \frac{2}{2x+y},\ -2y - \frac{1}{2x+y} \right\rangle = 0$$

$$f_{xx} = \frac{+4}{(2x+y)^2} \qquad f_{yy} = -2 + \frac{1}{(2x+y)^2}$$

$$f_{yx} = f_{xy} = \frac{2}{(2x+y)^2}$$

$$4 - \frac{2}{2x+y} = 0 \qquad 4(2x+y) = 2 \qquad y = \frac{1-4x}{2}$$

$$-2y - \frac{1}{2x+y} = 0 \qquad -2y(2x+y) = 1$$

$$-4x\left(\frac{1-4x}{2}\right) - 2\left(\frac{1-4x}{2}\right)^2 = 0$$

$$\frac{-4x+16x^2}{\cancel{2}} - \frac{1-8x+16x^2}{\cancel{2}} = 0$$

$$-4x - 1 + 8x = 0$$

$$\boxed{x = \frac{1}{4} \quad y = 0}$$

Discriminant

$$f_{xx} \cdot f_{yy} - [f_{xy}]^2$$

$$= \frac{4}{(2x+y)^2}\left(-2 + \frac{1}{(2x+y)^2}\right) - \left(\frac{2}{(2x+y)^2}\right)^2$$

@ $\left(\frac{1}{4}, 0\right)$

$$= \frac{4}{\left(\frac{1}{2}\right)^2}\left(-2 + \frac{1}{\left(\frac{1}{2}\right)^2}\right) - \left(\frac{2}{\left(\frac{1}{2}\right)^2}\right)^2$$

Cont on back

$$D = \frac{4}{\frac{1}{4}}\left(-2 + \frac{1}{\frac{1}{4}}\right) - \left(\frac{2}{\frac{1}{4}}\right)^2 = 16(2) - (8)^2$$

$$= 32 - 64 = -32$$

Since $D < 0$, $f\left(\frac{1}{4}, 0\right)$ is a <u>saddle point</u>

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad .$$

ans. $3.003\overline{3}$

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(\ldots)(y-b) + f_z(a\,b\,c)(z-c)$$

$$= \sqrt{2(1)^2+3(1)^2+(2)^2} + \frac{2x}{\sqrt{2x^2+3y^2+z^2}}(x-1) + \frac{3y}{\sqrt{2x^2+3y^2+z^2}}(y-1) + \frac{z}{\sqrt{2x^2+3y^2+z^2}}(z-2)$$

$\sqrt{2(1)^2+3(1)^2+(2)^2} = 3$

$$= 3 + \frac{2(1)}{3}(0.001) + \frac{3(1)}{3}(-0.001) + \frac{(2)}{3}(0.001)$$

$$= 3 + \frac{0.004}{3} - 0.001 = \frac{9 + 4E{-}3 - 3E{-}3}{3}$$

$$= \boxed{3.00\overline{333}}$$

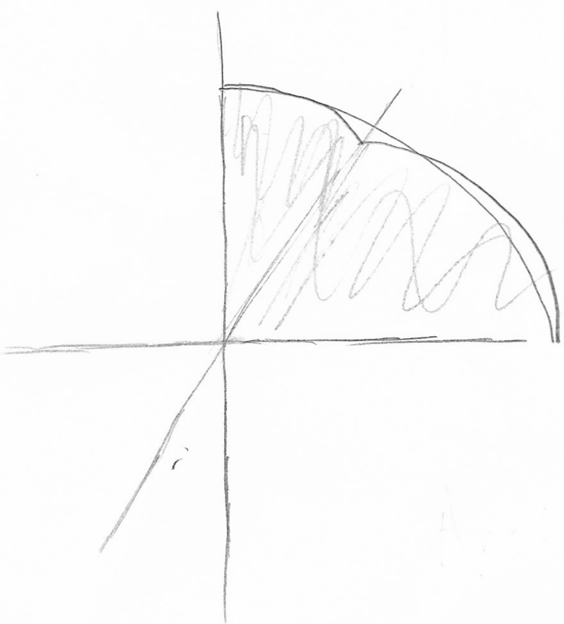
13. (12 points) Convert the triple iterated integral

$$\int_0^{2}\int_0^{\sqrt{4-z^2}}\int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates. Do not evaluate.**

---

ans. $\displaystyle\int_0^{\pi/2}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} \rho^6(\sin\phi\cos\theta)^2(\sin\phi\sin\theta)(\cos\phi)(\sin\phi)\,d\rho\, d\theta\, d\phi$



$x = \rho\sin\phi\cos\theta$

$y = \rho\sin\phi\sin\theta$

$z = \rho\cos\phi$

$dx\,dy\,dz = \rho^2\sin\phi\, d\rho\, d\theta\, d\phi$

$0 \le \rho \le 2$

$\frac{\pi}{2} \le \theta \le \pi$

$0 \le \phi \le \frac{\pi}{2}$

$$\text{Ans} = \int_0^{\pi/2}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\theta\, d\phi$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. 3

$$k(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$r'(t) = \langle 0, 3\cos(t), -3\sin(t)\rangle$$

$$r''(t) = \langle 0, -3\sin(t), -3\cos(t)\rangle$$

$$|r'(t) \times r''(t)| = \sqrt{(-9\cos^2(t) - 9\sin^2(t))^2 + 0^2 + 0^2}$$

$$= \sqrt{(-9(1))^2} = \boxed{9}$$

$$|r'(t)| = \sqrt{0 + (3\cos(t))^2 + (-3\sin(t))^2} = \sqrt{9(1)}$$

$$k = \frac{9}{3} = 3$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_0^v \sqrt{(2v^2-2u^2)^2-8u^2v}\; du\,dv$

$x=u^2 \quad y=uv \quad z=v^2$

$r_u = \langle 2u, v, 0\rangle$

$r_v = \langle 0, u, 2v\rangle$

$$r_u \times r_v = \det\begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = (2v^2-0)\hat{i} - (4uv-0)\hat{j} + (2u^2-0)\hat{k}$$

$$|r_u \times r_v| = \sqrt{4v^4-16u^2v^2+4u^4} = \sqrt{(2v^2-2u^2)^2-8u^2v^2}$$

$$\int_0^1 \int_0^v \sqrt{(2v^2-2u^2)^2-8u^2v^2}\; du\,dv$$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1, 1, 1)$.

---

ans. 15

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle$

$\nabla f \cdot \nabla g = y^2z^3 + 4xy^2z^3 + 9xy^2z^4$

@ $(1,1,1) = 1 + 4(1) + 9(1) = \boxed{15}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. 0

Plugging in obviously leads to $\frac{0}{0}$,

Approach along $y=mx$ & $w=cz$

$$\lim_{(x,z)\to(0,0)} \frac{(x+mx)^2-(z+cz)^2}{x+mx-(z+cz)} = \frac{x^2(1+m)^2-z^2(1+c)^2}{x(1+m)-z(1+c)}$$

Still $=\frac{0}{0}$ $\therefore$ go on

$z=ax$

$$\lim_{x\to 0} \frac{x^2(1+m^2)-(ax)^2(1+c)}{x(1+m)-(ax)(1+c)}$$

$$=\lim_{x\to 0} \frac{x^{\cancel{2}}\left((1+m)^2-a^2(1+c)^2\right)}{\cancel{x}\left((1+m)-a(1+c)\right)}$$

$$\lim_{x\to 0} x\left(\frac{(1+m)^2-a^2(1+c)^2}{(1+m)-a(1+c)}\right)=\boxed{0}$$