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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$
3. $z = \frac{-1}{12}x - \frac{5}{6}y + \frac{7\pi}{18}$
4. $3\hat{i} - 6\hat{j} + 9\hat{k}$
5. A: $\pi/3$ rad B: $\frac{2\pi}{3}$ rad C: $\pi/3$ rad
6. $-4\sqrt{3}$
7. $6e^{2u}(\cos^2(v)-\sin^2(v))$
8. 512π
9. -15
10. $(\frac{1}{4}, 0)$ is a saddle point
11. 3.0033
12. $\frac{\sqrt{2}}{6}$
13. $\longrightarrow$ $\int_0^{\pi/2}\int_{\pi/2}^{\pi}\int_0^{2} \rho^6(\sin\phi\cos\theta)^2(\sin\phi\sin\theta)(\cos\phi)(\sin\phi)\,d\rho\,d\theta\,d\phi$
14. 3
15. $\longrightarrow$ $\int_0^1\int_0^v \sqrt{(2v^2-2u^2)^2-8u^2v^2}\,du\,dv$
16. 15
17. 0

Sign the following declaration:

I [signature] Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: [signature]

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x,y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F}\cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (\cos(e^{\sin x}) + 5y)\, dx + (\sin(e^{\cos y}) + 11x)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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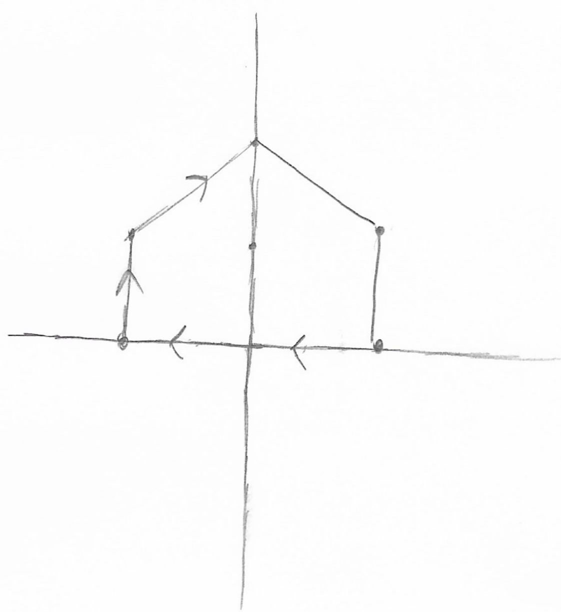
ans. Number: $\boxed{-18}$

C is closed, ∴ Green's Thm:

$$\iint_C \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$



$$= \int_{-1}^{0}\int_{0}^{x+2} (11-5)\, dy\, dx$$

$$+ \int_0^1\int_0^{2-x} 6\, dy\, dx = \int_{-1}^{0} 6(2+x)\, dx + \int_0^1 6(2-x)\, dx$$

$$= 12x + 3x^2 \Big|_{-1}^{0} + 12x - 3x^2 \Big|_0^1 = \underbrace{-(-12+3(1))}_{9} + \underbrace{(9-0)}_{9}$$

$$= 18, \text{ but CW, } \therefore = \boxed{-18}$$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x, y)\, dx\, dy \quad .$$

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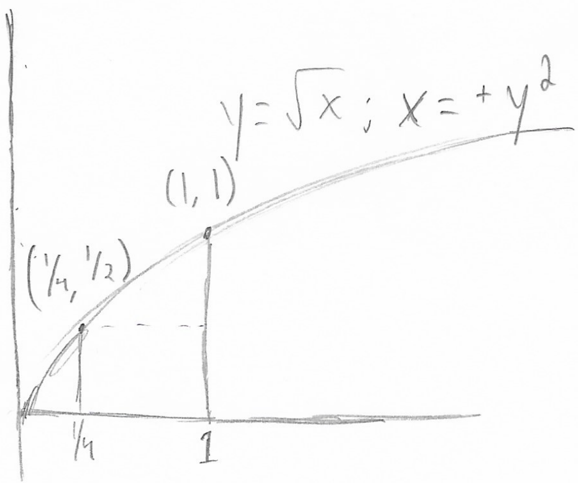
ans. Type I Integral : $\int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$



$$= \int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

ans. $z = \frac{-1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

$$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z)$$

$$\nabla f = \langle -2\sin(x+y) - 4\sin(x+z), -2\sin(x+y) - 8\sin(y+z), -4\sin(x+z) - 8\sin(y+z)\rangle$$

$$\nabla f(\tfrac{\pi}{6}, \tfrac{\pi}{6}, \tfrac{\pi}{6}) = \langle -2(\tfrac{\sqrt{3}}{2}) - 4(\tfrac{\sqrt{3}}{2}), -2(\tfrac{\sqrt{3}}{2}) - 8(\tfrac{\sqrt{3}}{2}), -4(\tfrac{\sqrt{3}}{2}) - 8(\tfrac{\sqrt{3}}{2})\rangle$$

$$= \langle -3\sqrt{3}, -5\sqrt{3}, -6\sqrt{3}\rangle$$

$$-3\cancel{\sqrt{3}}(x - \tfrac{\pi}{6}) - 5\cancel{\sqrt{3}}(y - \tfrac{\pi}{6}) - 6\cancel{\sqrt{3}}(z - \tfrac{\pi}{6}) = 0$$

$$-6z + \pi = 3(x - \tfrac{\pi}{6}) + 5(y - \tfrac{\pi}{6})$$

$$-6z = 3x - \frac{3\pi}{6} + 5y - \frac{5\pi}{6} - \frac{6\pi}{6}$$

$$z = \frac{3x}{-6} + \frac{5y}{-6} - \frac{\frac{14\pi}{6}}{-6} = \boxed{\frac{-1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}}$$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a}\times\mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k} \quad (1), \qquad \mathbf{b}\times\mathbf{c} = \mathbf{i}-\mathbf{j}+\mathbf{k} \quad (2), \qquad \mathbf{a}\times\mathbf{c} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k} \quad (3) .$$

What is

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\times(2\mathbf{a}-\mathbf{b}+3\mathbf{c}) \qquad ?$$

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ans. $3\hat{i} - 6\hat{j} + 9\hat{k}$

$$(a+b+c)\times(2a-b+3c)$$

$$= \cancel{(a\times 2a)} + \underbrace{(a\times -b)}_{-(1)} + \underbrace{(a\times 3c)}_{3(3)}$$
$$+ \underbrace{(b\times 2a)}_{-2(1)} + \cancel{(b\times -b)} + \underbrace{(b\times 3c)}_{3(2)}$$
$$+ \underbrace{(c\times 2a)}_{-2(3)} + \underbrace{(c\times -b)}_{(2)} + \cancel{(c\times 3c)}$$

$$= -(\hat{i}+\hat{j}-\hat{k}) + 3(2\hat{i}+\hat{j}+2\hat{k}) - 2(\hat{i}+\hat{j}-\hat{k})$$
$$+3(\hat{i}-\hat{j}+\hat{k}) - 2(2\hat{i}+\hat{j}+2\hat{k}) + (\hat{i}-\hat{j}+\hat{k})$$

$$= -3(\hat{i}+\hat{j}-\hat{k}) + 4(\hat{i}-\hat{j}+\hat{k}) + (2\hat{i}+\hat{j}+2\hat{k})$$

$$= \boxed{3\hat{i}-6\hat{j}+9\hat{k}}$$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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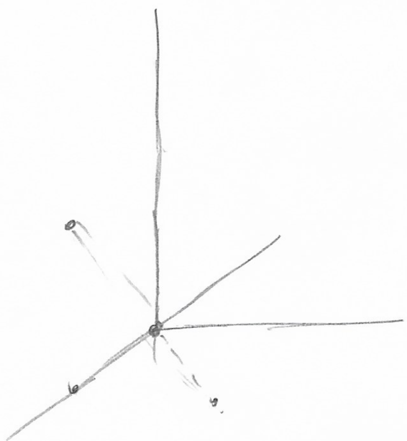
ans. The angle at A is: $\pi/3$ radians ;

The angle at B is: $2\pi/3$ radians ;

The angle at C is: $\pi/3$ radians ;



$$\vec{AB} = \langle 1, 0, 1 \rangle \quad |AB| = \sqrt{2}$$

$$\vec{AC} = \langle 1, 1, 0 \rangle \quad |AC| = \sqrt{2}$$

$$\vec{BC} = \langle 0, 1, -1 \rangle \quad |BC| = \sqrt{2}$$

$$\frac{\vec{AB} \cdot \vec{AC}}{|AB||AC|} = \frac{(1 \cdot 1 + 1 \cdot 0 + 1 \cdot 0)}{\sqrt{2}\ \sqrt{2}} = \frac{1}{2} = \cos\theta$$

$$\boxed{\theta_1 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$$

$$\frac{\vec{AB} \cdot \vec{BC}}{|AB||BC|} = \frac{(1 \cdot 0 + 0 \cdot 1 + (-1 \cdot 1))}{\sqrt{2}\ \sqrt{2}} = \frac{-1}{2}$$

$$\boxed{\theta_2 = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}}$$

$$\frac{\vec{AC} \cdot \vec{BC}}{|AC||BC|} = \frac{(1 \cdot 0 + 1 \cdot 1 + (0 \cdot -1))}{\sqrt{2}\ \sqrt{2}} = \frac{1}{2}$$

$$\boxed{\theta_3 = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}}$$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

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ans. $-4\sqrt{3}$

$$\nabla f = \langle 3x^2 + yz,\ 3y^2 + xz,\ 3z^2 + xy \rangle$$

$$\vec{u} = \frac{1}{\sqrt{3}} \langle -1, -1, -1 \rangle = \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$$

$$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$$

$$\nabla f \cdot \vec{u} = \left(\frac{-4}{\sqrt{3}}\right) + \left(\frac{-4}{\sqrt{3}}\right) + \left(\frac{-4}{\sqrt{3}}\right) = \frac{-12}{\sqrt{3}} = -4\sqrt{3}$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain!

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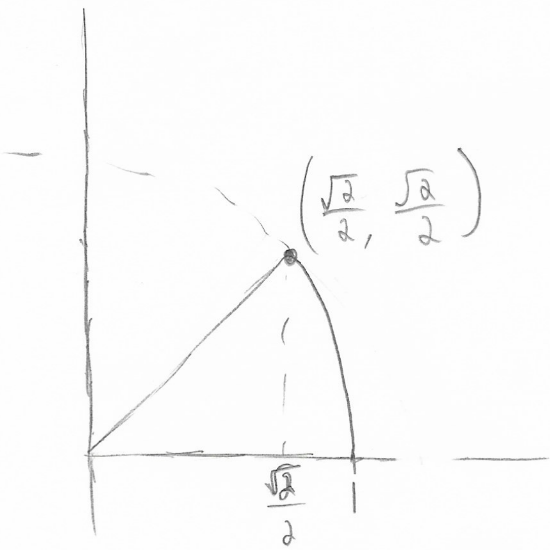
ans. $\frac{\sqrt{2}}{6}$



$X = r\cos\Theta$

$0 \le r \le 1$

$0 \le \Theta \le \frac{\pi}{4}$

$$= \int_0^1 \int_0^{\pi/4} r\cos\Theta \cdot r \, d\Theta \, dr$$

$$= \int_0^1 r^2\left(\sin\left(\tfrac{\pi}{4}\right) - 0\right) dr = \frac{\sqrt{2}}{2}\int_0^1 r^2 \, dr = \boxed{\frac{\sqrt{2}}{6}}$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

ans. $6\, e^{2u}(\cos^2(v) - \sin^2(v))$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$$

$$= 6x \cdot e^u \cos(v) + (-6y)\, e^u \sin v$$

$$= 6(e^u \cos(v))(e^u \cos v) - 6(e^{2u} \sin^2 v)$$

$$= \boxed{6\, e^{2u}(\cos^2(v) - \sin^2(v))}$$