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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

1.

2. 7/12

3. $z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{4\pi\sqrt{3}}{6}$

4. $\langle -1, -8, 5 \rangle$

5. $\theta_A = \frac{2\pi}{3}$, $\theta_B = \frac{2\pi}{3}$, $\theta_C = \frac{2\pi}{3}$

6. $-5\sqrt{3}$

7. -1.8

8.

9.

10. $(0, 1/2)$: inconclusive

11.

12.

13.

14. 3

15. $\int_0^1 \int_0^u r(u,v)\, dv\, du$

16. 14

17. DNE

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\, dx \, + \, (sin\,(e^{\cos y}) \, + \, 11x\,)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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ans.

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2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\, dy\, dx$$

---

ans. $7/12$

$$\int_{0}^{\sqrt{x}}\int_{\frac{1}{4}}^{1} f(x,y)\, dx\, dy$$

$$\Rightarrow \int_{1/4}^{1} \sqrt{x}\, dx = \int_{1/4}^{1} x^{1/2}\, dx$$

$$\Rightarrow \frac{2}{3} x^{3/2} \Big|_{1/4}^{1} \boxed{= \frac{7}{12}}$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \;=\; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -3\sqrt{3}\,x - 5\sqrt{3}\,y + \frac{9\pi\sqrt{3}}{6}$

$f_x = -2\sin(x+y) - 4\sin(x+z) \Rightarrow f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -3\sqrt{3}$

$f_y = -2\sin(x+y) - 8\sin(y+z) \Rightarrow f_y(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -5\sqrt{3}$

$z - \frac{\pi}{6} = -3\sqrt{3}\left(x - \frac{\pi}{6}\right) - 5\sqrt{3}\left(y - \frac{\pi}{6}\right)$

$z = -3\sqrt{3}\,x - 5\sqrt{3}\,y + \frac{5\pi\sqrt{3}}{6} + \frac{\pi\sqrt{3}}{2} + \frac{\pi}{6}$

$z = -3\sqrt{3}\,x - 5\sqrt{3}\,y + \frac{9\pi\sqrt{3}}{6}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $\langle -1, -8, 5 \rangle$

$$C = \begin{vmatrix} i & j & k \\ a & b & c \\ 2a & -b & 3c \end{vmatrix} \Rightarrow [3cb - -bc]\, i - [3ca - 2ac]\, j + [-ba - 2ab]\, k$$

$$\Rightarrow [3i - 3j + 3k + i - j + k] - [6i + 3j + 6k - 4i - 2j - 4k] + [-i - j + k - 2i - 2j + 2k]$$

$$\Rightarrow [4i - 4j + 4k] - [2i + j + 2k] + [-3i - 3j + 3k]$$

$$= -i - 8j + 5k$$

$AB = \langle 1, 0, 1 \rangle$
$BC = \langle 0, 1, -1 \rangle$
$CA = \langle -1, -1, 0 \rangle$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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ans. The angle at A is: $\frac{2\pi}{3}$ radians ;

The angle at B is: $\frac{2\pi}{3}$ radians ;

The angle at C is: $\frac{2\pi}{3}$ radians ;

$\cos\theta = (A_x B_x + A_y B_y + A_z B_z) / (\|A\| \|B\|)$ ← Ex. (A & B Arbitrary)

$\cos\theta_A = (1 \cdot -1 + 0 \cdot -1 + 1 \cdot 0) / ((\sqrt{2})(\sqrt{2}))$

$\theta_A = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

$\cos\theta_B = (1 \cdot 0 + 1 \cdot 0 + 1 \cdot -1) / ((\sqrt{2})(\sqrt{2}))$

$\theta_B = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

$\cos\theta_C = (0 \cdot -1 + 1 \cdot -1 + -1 \cdot 0) / ((\sqrt{2})(\sqrt{2})$

$\theta_C = \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans. $-5\sqrt{3}$

$$\frac{\partial f}{\partial x} = 3x^2 + yz \quad \frac{\partial f}{\partial y} = 3y^2 + xz \quad \frac{\partial f}{\partial z} = 3z^2 + xy$$

$$\frac{\partial f}{\partial x}(1,1,1) = 5 \quad \frac{\partial f}{\partial y}(1,1,1) = 5 \quad \frac{\partial f}{\partial x}(1,1,1) = 5$$

$$\nabla f(1,1,1) = 5i + 5j + 5k$$

$$u = 5u_1 + 5u_2 + 5u_3 \Rightarrow u = \frac{(-1,-1,-1)}{\|(-1,-1,-1)\|} = \left(-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}\right)$$

$$\partial_u f(1,1,1) = \frac{-5\sqrt{3}}{3} - \frac{5\sqrt{3}}{3} - \frac{5\sqrt{3}}{3} = \boxed{-5\sqrt{3}}$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. -1.8

$$g(u,v) = 3(e^u \cos v)^2 - 3(e^u \sin v)^2$$

$$\frac{\partial g}{\partial u} = 6(e^u \cos v) - 6(e^u \sin v)$$

$$= 6(1 e^u \cos v) - 6(1 e^u \sin v)$$

$$\frac{\partial g}{\partial u}(0,1) = 6\cos(1) - 6\sin(1)$$

$$= -1.8$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans.

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9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} \, = \, \langle \, 3z \, , \, 2x \, , \, y+z \, \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans.

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10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. C.P: $(0, \frac{1}{2})$ inconclusive

$f_x = 4 - \frac{2}{2x+y}$ $\quad$ $f_y = 4 - \frac{1}{2x+y}$ $\quad$ $f_{xx} = \frac{4}{(2x+y)^2}$

$4 - \frac{2}{2x+y} = 0 \Rightarrow 8x + 4y = 2$

$\Rightarrow y = \frac{1}{2} - 2x$

$f_{xy} = \frac{2}{(2x+y)^2}$

$4 - \frac{1}{2x+y} = 0$

$\Rightarrow 4 - \frac{1}{2x + \frac{1}{2} - 2x}$

$f_{yy} = \frac{1}{(2x+y^2)}$

$x = 0 \quad y = \frac{1}{2}$

$f_{xx}(0, 1/2) = 16$

$D = 16 \cdot 4 - 8^2 = 0$ inconclusive

$f_{xy}(0, 1/2) = 8$

$f_{yy}(0, \frac{1}{2}) = 4$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) \;=\; \sqrt{2x^2+3y^2+z^2} \quad .$$

---

ans.

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12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx \; + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain!

---

ans.

---

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans.

$r = \sqrt{x^2+y^2+z^2}$

$\theta = \cos^{-1}\left(\frac{z}{r}\right)$

$\phi = \tan^{-1}\left(\frac{y}{x}\right)$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. 3

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix}$

$\Rightarrow -9\cos^2 t - 9\sin^2 t\, i - 0 - 0\, j + 0 - 0\, k$

$r'(t) \times r''(t) = -9\cos^2 t - 9\sin^2 t$

$\Rightarrow t = \frac{\pi}{3}$

$\Rightarrow \frac{|r'(t) \times r''(t)|}{r'(t)} = \frac{9}{3} \boxed{= 3}$

$r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3}) = -9$

$|r'(\frac{\pi}{3})| = \sqrt{0^2 + (\frac{3}{2})^2 + (\frac{-3\sqrt{3}}{2})^2} = 3$

$|r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3})| = \sqrt{81} = 9$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_0^u r(u,v) \, dv \, du$

$u < v$

$0 < u$

$v < 1$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \quad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle ; \nabla g = \langle 1, 2y, 3z^2 \rangle$

$\nabla f \cdot \nabla g = y^2z^3 + 4xy^2z^3 + 9xy^2z^4$

$\nabla f \cdot \nabla g(1,1,1) = 1 + 4 + 9 \boxed{= 14}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

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ans. DNE

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(0+0)^2-(0+0)^2}{0+0-0-0} = \text{inconclusive}$$

$\Rightarrow y = cx$

$$\Rightarrow \lim_{(x,y,z,w)\to(0,cx,0,0)} \frac{(0+cx)^2-(0+0)^2}{0+cx-0-0} = \text{DNE}$$

Slope depends on c