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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 18
2. $\int_0^1 \int_{\frac{e^y}{4}}^{1} f(x,y)\, dx\, dy$
3. $-\frac{1}{2}x - \frac{5}{6}y + \frac{14\pi}{36}$
4. 0
5. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{2\pi}{3}$
6. $-\frac{8\sqrt{12}}{3}$
7. $6(\cos^2(1) - \sin^2(1))$
8. 18π
9. -15
10. local mins at $(-\frac{1}{4}, -\frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$
11. 3.00
12. $\frac{\sqrt{2}}{6}$
13. $\frac{128}{105}$
14. $3/10$
15. $\int_0^1 \int_1^u -2u^3 - u^2v + v^2 \; dv\; du$
16. 14
17. 0

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

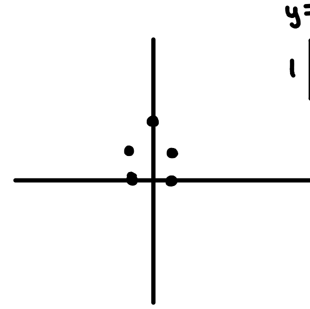
---

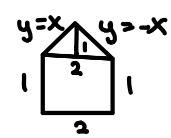
ans. 18

$P = \cos(e^{\sin x}) + 5y$ $\qquad Q = \sin(e^{\cos y}) + 11x$

$\frac{\partial P}{\partial y} = 5$ $\qquad \frac{\partial Q}{\partial x} = 11$

$11 - 5 = 6$





$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$

$= \iint_D 6 \; dA$

Area $= 2(1) + \frac{1}{2}(2)(1) = 2 + 1 = 3$

$6(3) = 18$

If the intregrand is a constant number, you can take it out and multiply by the volume.

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, d\mathbf{y}\, d\mathbf{x} \qquad .$$

---

ans. $\displaystyle\int_0^1 \int_{\frac{e^y}{4}}^{1} f(x,y)\, dx\, dy$

$D: \{ (x,y) \mid \frac{1}{4} < x < 1 \ , \ 0 \leq y \leq \sqrt{x} \}$

$D: \{ (x,y) \mid 0 \leq y \leq 1 \ , \ \frac{e^y}{4} \leq x \leq 1 \}$

$$\int_0^1 \int_{\frac{e^y}{4}}^{1} f(x,y)\, dx\, dy$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \ = \ 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{14\pi}{36}$

$$-2\sin(x+y) + \left(-4\sin(x+z) - 4\sin(x+z)\frac{\partial z}{\partial x}\right) - 8\sin(y+z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{2\sin(x+y) + 4\sin(x+z)}{-4\sin(x+z) - 8\sin(y+z)}$$

$$-2\sin(x+y) - 4\sin(x+z)\frac{\partial z}{\partial y} + \left(-8\sin(y+z) - 8\sin(y+z)\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial z}{\partial y} = \frac{2\sin(x+y) + 8\sin(y+z)}{-4\sin(x+z) - 8\sin(y+z)}$$

$$\frac{\partial z}{\partial x}\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{2\sin(\frac{\pi}{3}) + 4\sin(\frac{\pi}{3})}{-4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})} = \frac{\sqrt{3} + 2\sqrt{3}}{-2\sqrt{3} - 4\sqrt{3}} = \frac{3\sqrt{3}}{-6\sqrt{3}} = -\frac{1}{2}$$

$$\frac{\partial z}{\partial y}\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{2\sin(\frac{\pi}{3}) + 8\sin(\frac{\pi}{3})}{-4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})} = \frac{\sqrt{3} + 4\sqrt{3}}{-2\sqrt{3} - 4\sqrt{3}} = \frac{5\sqrt{3}}{-6\sqrt{3}} = -\frac{5}{6}$$

$$z - \frac{\pi}{6} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{5}{6}\left(y - \frac{\pi}{6}\right)$$

$$z = -\frac{1}{2}x + \frac{\pi}{12} - \frac{5}{6}y + \frac{5\pi}{36} + \frac{\pi}{6}$$

$$z = -\frac{1}{2}x + \frac{3\pi}{36} - \frac{5}{6}y + \frac{5\pi}{36} + \frac{6\pi}{36}$$

$$z = -\frac{1}{2}x - \frac{5}{6}y + \frac{14\pi}{36}$$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

$\langle 1,1,-1\rangle$ $\quad$ $\langle 1,-1,1\rangle$ $\quad$ $\langle 2,1,2\rangle$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. 0

---

cross prod = 0

$\downarrow$

perpendicular

$$\begin{vmatrix} i & j & k \\ a_x+b_x+c_x & a_y+b_y+c_y & a_z+b_z+c_z \\ 2a_x-b_x+3c_x & 2a_y-b_y+3c_y & 2a_z-b_z+3c_z \end{vmatrix}$$

$$= i\big((a_y+b_y+c_y)(2a_z-b_z+3c_z) - (a_z+b_z+c_z)(2a_y-b_y+3c_y)\big)$$

$$-j\big((a_x+b_x+c_x)(2a_z-b_z+3c_z) - (a_z+b_z+c_z)(2a_x-b_x+3c_x)\big)$$

$$+k\big((a_x+b_x+c_x)(2a_y-b_y+3c_y) - (a_y+b_y+c_y)(2a_x-b_x+3c_x)\big)$$

$$\langle 1,1,-1\rangle + \langle 1,-1,1\rangle + \langle 2,1,2\rangle$$

$$= \langle 4,1,2\rangle$$

$$\begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i(a_yb_z - a_zb_y) - j(a_xb_z - a_zb_x) + k(a_xb_y - a_yb_x)$$

$\overset{1}{a_y}\overset{2}{b_z} - \overset{1}{a_z}\overset{1}{b_y} = 1 \qquad \overset{1}{a_x}\overset{2}{b_z} - \overset{1}{a_z}\overset{3}{b_x} = -1 \qquad \overset{1}{a_x}\overset{2}{b_y} - \overset{1}{a_y}\overset{3}{b_x} = -1$

$$\begin{vmatrix} i & j & k \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = i(b_yc_z - b_zc_y) - j(b_xc_z - b_zc_x) + k(b_xc_y - b_yc_x)$$

$\overset{1}{b_y}\overset{3}{c_z} - \overset{2}{b_z}\overset{1}{c_y} = 1 \qquad \overset{3}{b_x}\overset{3}{c_z} - \overset{2}{b_z}\overset{4}{c_x} = 1 \qquad \overset{3}{b_x}\overset{1}{c_y} - \overset{1}{b_y}\overset{4}{c_x} = 1$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{2\pi}{3}$ radians ;

The angle at C is: $\frac{2\pi}{3}$ radians ;

$AB = \langle 1-0, 0-0, 1-0 \rangle = \langle 1, 0, 1 \rangle \qquad \sqrt{1+0+1} = \sqrt{2}$

$BC = \langle 1-1, 1-0, 0-1 \rangle = \langle 0, 1, -1 \rangle \qquad \sqrt{0+1+1} = \sqrt{2}$

$CA = \langle 0-1, 0-1, 0-0 \rangle = \langle -1, -1, 0 \rangle \qquad \sqrt{1+1+0} = \sqrt{2}$

$AB \cdot BC = 0 + 0 - 1 = -1$

$BC \cdot CA = 0 - 1 + 0 = -1$

$CA \cdot AB = -1 + 0 + 0 = -1$

$$\cos\theta = \frac{AB \cdot BC}{|AB|\,|BC|} = \frac{1}{\sqrt{2}\,(\sqrt{2})} = \frac{1}{2} \qquad \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\cos\theta = \frac{BC \cdot CA}{|BC|\,|CA|} = \frac{-1}{\sqrt{2}\,(\sqrt{2})} = \frac{-1}{2} \qquad \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$

$$\cos\theta = \frac{CA \cdot AB}{|CA|\,|AB|} = \frac{-1}{\sqrt{2}(\sqrt{2})} = -\frac{1}{2} \qquad \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $\overset{P}{(1, 1, 1)}$ in a direction pointing to the point $\overset{Q}{(-1, -1, -1)}$.

---

ans. $-\frac{8\sqrt{12}}{3}$

$PQ = \langle -2, -2, -2 \rangle$

$\sqrt{4+4+4} = \sqrt{12}$

$u = \langle \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}} \rangle$

$\nabla f = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle$

$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$

$\nabla f(1,1,1) \cdot u$

$= -\frac{8}{\sqrt{12}} - \frac{8}{\sqrt{12}} - \frac{8}{\sqrt{12}} = -\frac{32}{\sqrt{12}} = -\frac{32\sqrt{12}}{12} = -\frac{8\sqrt{12}}{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\,(\cos^2(1) - \sin^2(1))$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\left(\frac{\partial x}{\partial u}\right) + \frac{\partial g}{\partial y}\left(\frac{\partial y}{\partial u}\right)$$

$$\frac{\partial g}{\partial x} = 6x \qquad \frac{\partial g}{\partial y} = -6y$$

$$\frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial g}{\partial u} = 6x\,(e^u \cos v) - 6y\,(e^u \sin v)$$

$$= 6e^u \cos v\, e^u \cos v - 6e^u \sin v\, e^u \sin v$$

$$\frac{\partial g}{\partial u}(0,1) = 6e^0 \cos 1\, e^0 \cos 1 - 6e^0 \sin 1\, e^0 \sin 1$$

$$= 6\cos^2(1) - 6\sin^2(1)$$

$$= 6\,(\cos^2(1) - \sin^2(1))$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

4th quad

---

ans. 18π

$r^2 + z^2 < 4$

$z^2 \leq 4 - r^2$

$r^2 = 4$

$r = 2$

$\text{div}(F) = 3 - 2 + 5 = 6$

$\iiint_E 6\, dV = 3 \iiint dV = 3 \times \text{volume}$

$D : \{ (r, \theta, z) \mid 0 \leq r \leq 2,\ -\frac{\pi}{2} \leq \theta \leq \pi,\ 0 \leq z \leq 4 - r^2 \}$

$$3 \int_0^2 \int_{-\frac{\pi}{2}}^{\pi} \int_0^{4-r^2} 1 \; dz\, d\theta\, r dr$$

$$\int_0^{4-r^2} r\, dz \to rz \Big|_0^{4-r^2} = r(4 - r^2) = 4r - r^3$$

$$\int_{-\frac{\pi}{2}}^{\pi} 4r - r^3\, d\theta = 4r\theta - r^3\theta \Big|_{-\frac{\pi}{2}}^{\pi} = 4\pi r - \pi r^3 - \left(-2\pi r + \frac{\pi r^3}{2}\right)$$

$$= 6\pi r - \frac{3\pi r^3}{2}$$

$$3 \int_0^2 6\pi r - \frac{3r^3\pi}{2}\, dr \to 3\pi r^2 - \frac{3\pi r^4}{8} \Big|_0^2 = 12\pi - 6\pi = 6\pi$$

$3(6\pi) = 18\pi$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3z\,,\, 2x\,,\, y+z \,\rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

$+$

with **upward pointing** normal.

---

ans. -15

$g = 2x+3y$

$g_x = 2 \quad g_y = 3$

$P = 3z \quad Q = 2x \quad R = y+z$

$$\iint_S F \cdot dS = \iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA$$

$$= \iint_D (-3z(2) - 2x(3) + y+z)\, dA$$

$$= \iint_D (-6(2x+3y) - 6x + y + 2x+3y)\, dA$$

$$= \iint_D -12x - 18y - 6x + y + 2x + 3y \quad dA$$

$$= \int_0^1 \int_0^1 -16x - 14y \quad dx\, dy$$

$$\frac{-16x^2}{2} - 14xy \Big|_0^1 = -8 - 14y$$

$$\int -8 - 14y \; dy$$

$$-8y - \frac{14y^2}{2}\Big|_0^1 = -8 - 7 = -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. local mins at $(-\frac{1}{4}, -\frac{1}{2})$ and $(\frac{1}{2}, -\frac{1}{2})$

$f_x = 4 - \frac{2}{2x+y}$

$4 - \frac{2}{2x+y} = 0$

$\frac{-2}{2x+y} = -4$

$\frac{-2}{-4} = 2x+y$

$\frac{1}{2} = 2x+y$

$\frac{1}{2} = 2x - \frac{1}{2}$

$2x = 1$

$x = \frac{1}{2}$

$(\frac{1}{2}, -\frac{1}{2})$

$f_y = 2y - \frac{1}{2x+y}$

$2y - \frac{1}{2x+y} = 0$

$-1 = 2y(2x+y)$

$2y = -1$

$y = -\frac{1}{2}$

$2x + y = -1$

$2x - \frac{1}{2} = -1$

$2x = -\frac{1}{2}$

$x = -\frac{1}{4}$

$(-\frac{1}{4}, -\frac{1}{2})$

$f_{xx} = \frac{4}{(2x+y)^2}$

$f_{xy} = \frac{2}{(2x+y)^2}$

$f_{yy} = 2 + \frac{2}{(2x+y)^2}$

when domain is positive and fxx is positive, the point is a local minimum

$f_{xx} = \frac{4}{(1-\frac{1}{2})^2} = \frac{4}{\frac{1}{4}} = 16$

$f_{xy} = \frac{2}{(1-\frac{1}{2})^2} = \frac{2}{\frac{1}{4}} = 8$

$f_{yy} = 2 + \frac{2}{(1-\frac{1}{2})^2} = 2+8 = 10$

$D = 16(10) - (8)^2 = 94$

min since D is + and fxx is +

$f_{xx} = \frac{4}{(-\frac{1}{2}-\frac{1}{2})^2} = \frac{4}{1} = 4$

$f_{xy} = \frac{2}{(-\frac{1}{2}-\frac{1}{2})^2} = \frac{2}{1} = 2$

$f_{yy} = 2 + \frac{2}{(-\frac{1}{2}-\frac{1}{2})^2} = 2+2 = 4$

$D = 4(4) - (2)^2 = 12$

min since D is + and fxx is +

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.00

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nice point → $(1,1,2)$

$f(1,1,2) = \sqrt{2+3+4} = \sqrt{9} = 3$

$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x \rightarrow @\,(1,1,2) : \dfrac{4}{2\sqrt{2+3+4}} = \dfrac{4}{2\sqrt{9}} = \dfrac{4}{6} = \dfrac{2}{3}$

$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y \rightarrow (1,1,2) = \dfrac{6}{2\sqrt{2+3+4}} = \dfrac{6}{2(3)} = 1$

$f_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 2z \rightarrow (1,1,2) = \dfrac{4}{2\sqrt{2+3+4}} = \dfrac{4}{2(3)} = \dfrac{4}{6} = \dfrac{2}{3}$

$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$

$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(0.001) + 1(-0.001) + \frac{2}{3}(0.001)$

$= 3 + \dfrac{0.002}{3} - 0.001 + \dfrac{0.002}{3}$

$\approx 3.0000\bar{3}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $\frac{\sqrt{2}}{6}$

$D: \{(x,y) \mid 0 \le x \le \frac{\sqrt{2}}{2},\ 0 \le y \le x\} \cup \{(x,y) \mid \frac{\sqrt{2}}{2} \le x \le 1,\ 0 \le y \le \sqrt{1-x^2}\}$

$D: \{(r,\theta) \mid 0 \le r \le \frac{\sqrt{2}}{2},\ 0 \le \theta \le \frac{\pi}{2}\} \cup \{(r,\theta) \mid 0 \le r \le \underset{=}{1-\frac{\sqrt{2}}{2}},\ 0 \le \theta \le \pi\}$

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\sqrt{2}}{2}} r\cos\theta \ r\,dr\,d\theta + \int_0^{\pi} \int_0^{\frac{2-\sqrt{2}}{2}} r\cos\theta \ r\,dr\,d\theta$$

$$\frac{r^3\cos\theta}{3}\Bigg|_0^{\frac{\sqrt{2}}{2}} \qquad\qquad \frac{r^3\cos\theta}{3}\Bigg|_0^{\frac{2-\sqrt{2}}{2}}$$

$$\int_0^{\frac{\pi}{2}} \frac{4\sqrt{2}}{24}\cos\theta \qquad\qquad \int_0^{\pi} \frac{10-7\sqrt{2}}{12}\cos\theta$$

$$= \frac{4\sqrt{2}}{24}\sin\theta\Bigg|_0^{\frac{\pi}{2}} + \frac{10-7\sqrt{2}}{12}\sin\theta\Bigg|_0^{\pi}$$

$$= \frac{4\sqrt{2}}{24} = \frac{\sqrt{2}}{6} \qquad\qquad = 0$$

$$\frac{\sqrt{2}}{6} + 0 = \frac{\sqrt{2}}{6}$$

Both are the same graph with different ranges so adding together gives the answer.

13. (12 points) Convert the triple iterated integral

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^0 x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\frac{128}{105}$

$x^2+y^2+z^2=\rho^2$

$D: \{ (x,y,z) \mid \underbrace{-\cancel{\#} \le x \le 0 \,,\, 0 \le y \le +\cancel{\#}}_{\text{2nd quad}} \,,\, 0 \le z \le 2 \}$

$dV = dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho\cos\phi$

$$\int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \int_0^2 (\rho\sin\phi\cos\theta)^2 (\rho\sin\phi\sin\theta)(\rho\cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$$

$$\int_{\frac{\pi}{2}}^{\pi} \cos^2\theta \sin\theta\, d\theta \quad \int_{\frac{\pi}{2}}^{\pi} \underbrace{\sin^2\phi \sin\phi \cos\phi \sin\phi}_{\sin^4\phi\cos\phi}\, d\phi \quad \int_0^2 \underbrace{\rho^2(\rho)(\rho)(\rho^2)}_{\rho^6}\, d\rho$$

$$-\frac{1}{3}\cos^3\theta \Big|_{\frac{\pi}{2}}^{\pi} \qquad \frac{1}{5}\sin^5\theta \Big|_{\frac{\pi}{2}}^{\pi} \qquad \frac{\rho^7}{7}\Big|_0^2$$

$$\left(-\frac{1}{3}\cos^2\pi + \frac{1}{3}\cos^2\frac{\pi}{2}\right)\left(\frac{1}{5}\sin^5\pi - \frac{1}{5}\sin^5\frac{\pi}{2}\right)\left(\frac{2^7}{7}\right)$$

$$= -\frac{1}{3}\left(-\frac{1}{5}\right)\left(\frac{128}{7}\right) = \frac{128}{105}$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{3}{10}$

$r'(t) = \langle 1, 3\cos t, -3\sin t \rangle$

$r'(\frac{\pi}{3}) = \langle 1, 3\cos\frac{\pi}{3}, -3\sin\frac{\pi}{3} \rangle = \langle 1, \frac{3}{2}, \frac{-3\sqrt{3}}{2} \rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$r''(\frac{\pi}{3}) = \langle 0, -3\sin\frac{\pi}{3}, -3\cos\frac{\pi}{3} \rangle = \langle 0, \frac{-3\sqrt{3}}{2}, -\frac{3}{2} \rangle$

$|r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3})| =$

$$\begin{vmatrix} i & j & k \\ 1 & \frac{3}{2} & \frac{-3\sqrt{3}}{2} \\ 0 & \frac{-3\sqrt{3}}{2} & -\frac{3}{2} \end{vmatrix} = i\left(\frac{3}{2}\left(-\frac{3}{2}\right) - \left(\frac{-3\sqrt{3}}{2}\right)\left(\frac{-3\sqrt{3}}{2}\right)\right) - j\left(-\frac{3}{2} - 0\right) + k\left(\frac{-3\sqrt{3}}{2} - 0\right)$$

$-\frac{9}{4} - \frac{27}{4} = -\frac{36}{4} = -9$

$\langle -9, \frac{3}{2}, \frac{-3\sqrt{3}}{2} \rangle$

$\sqrt{81 + \frac{9}{4} + \frac{27}{4}} = \sqrt{90}$

$|r'(\frac{\pi}{3})| = \sqrt{1 + \frac{9}{4} + \frac{27}{4}} = (\sqrt{10})^3 = 10\sqrt{10}$

$\frac{\sqrt{90}}{10\sqrt{10}} = \frac{\sqrt{900}}{100} = \frac{30}{100} = \frac{3}{10}$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_1^u -2u^3 - u^2 v + v^2 \;\; dv \;\; du$

$P = u^2 \qquad Q = uv \qquad R = v^2$

$P_u = 2u \qquad Q_u = v \qquad R_u = 0$

$P_v = 0 \qquad Q_v = u \qquad R_v = 2v$

$D: \{ (u,v) \mid 0 \le u \le 1 \;,\; u \le v \le 1 \}$

$\int_0^1 \int_1^u -u^2(2u) - uv(u) + v^2 \;\; dv \;\; du$

$\int_0^1 \int_1^u -2u^3 - u^2 v + v^2 \;\; dv \;\; du$

16. (12 points) Let

$$f(x,y,z) \;=\; xy^2z^3 \quad ,$$

and let

$$g(x,y,z) \;=\; x+y^2+z^3 \quad .$$

compute the dot-product

$$grad(f)\,.\,grad(g) \qquad .$$

at the point $(1,1,1)$.

---

ans. 14

$grad(f) = \langle y^2z^3 ,\ 2xyz^3 ,\ 3xy^2z^2 \rangle$

$grad(f)(1,1,1) = \langle 1, 2, 3 \rangle$

$grad(g) = \langle 1 ,\ 2y ,\ 3z^3 \rangle$

$grad(g)(1,1,1) = \langle 1, 2, 3 \rangle$

$\langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle$

$= 1+4+9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. 0

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{0-0}{0+0-0-0} = \frac{0}{0}$$

let $y, z, w = 0$

$$\lim_{x\to 0} \frac{x^2}{x} = \lim_{x\to 0} x = 0$$

same → limit exists $= 0$

let $x, y, z = 0$

$$\lim_{w\to 0} \frac{-w^2}{-w} = \lim_{w\to 0} w = 0$$