NAME: (print!) Khush Tated RUID: (print!) 192002457

SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as finalFirstLast.pdf to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 18.

2. $\int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy + \int_{0}^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy$

3. $-\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18} = z$

4. $\langle 3, -6, 9 \rangle$

5. $A, B, C = \frac{\pi}{3}$

6. $-4\sqrt{3}$

7. $6\cos^2(1) - 6\sin^2(1)$.

8. 8π.

9. -15.

10. $(\frac{3}{4}, -1)$ is a saddle point.

11. About 3.0003.

12. idk pi did work

13. $\int_0^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} (\rho\sin\phi\cos\theta)^2 (\rho\sin\phi)(\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\,d\theta\, d\phi\, d\rho$

14. $\frac{1}{3}$

15.

16. 14.

17. 0

Sign the following declaration: Khush

I   Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the vector-field line integral

$$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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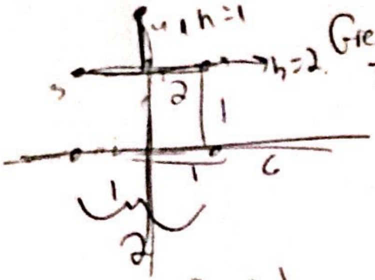
ans. 18



Green's Theorem

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) dA$$

$(2\cdot 1)$

$$P = \cos \cdot e^{\sin x} + 5y$$

$$P_y = 5$$

$$Q_x = 11$$

$$11 - 5 = 6$$

$$\iint_D 6\,dA$$

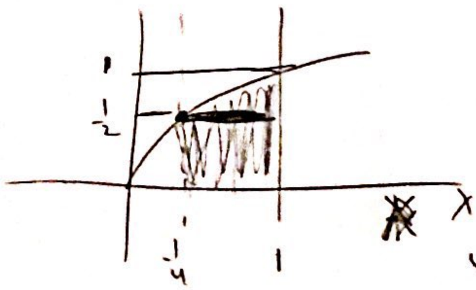
$$6 \cdot (1 + 2) = 18.$$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,\cancel{dx\,dy}\;dy\,dx.$$

---

ans. $\displaystyle \int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy \;+\; \int_{0}^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy$



$\sqrt{\frac{1}{4}}$.

$x:[0,y^2]$ y doesn't work

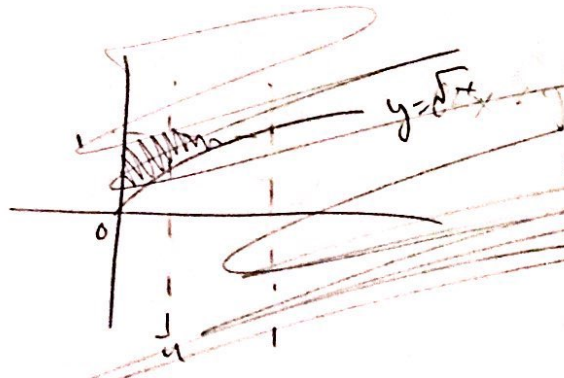
$X=y^2$

$0<x<y^2$

y:

$$\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$$

$$+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy$$



$y \to [0,1]$

$x \to [0,y^2]$.

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3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18} = z$

$0 = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x = -2\sin(x+y) - 4\sin(x+z)$ at $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -3\sqrt{3}$

$f_y = -2\sin(x+y) - 8\sin(y+z) = -5\sqrt{3}$

$f_z = -4\sin(x+z) - 8\sin(y+z) = 6\sqrt{3}$.

$-3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6}) = 0$

$-3(x - \frac{\pi}{6}) - 5(y - \frac{\pi}{6}) - 6(z - \frac{\pi}{6}) = 0$.

$\frac{14}{6} = \frac{7}{3}$ $\qquad -3x + \frac{\pi}{2} - 5y + \frac{5\pi}{6} - 6z + \pi = 0$. $\qquad \frac{3\pi}{6} + \frac{5\pi}{6}$

$\frac{6\pi}{6} +$

$-3x - 5y + \frac{3\pi}{6} + \frac{5\pi}{6} + \frac{6\pi}{6} = 6z$

$\dfrac{-3x - 5y + \frac{7\pi}{3}}{6} = \frac{6z}{6}$

$-\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18} = z$

22 . 2.

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $\langle 3, -6, 9 \rangle$

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0 ✓ ✓ ✓ x ✓

$a \times 2a + a \times -b + a \times 3c + b \times 2a + b \times -b + b \times 3c$

$+ c \times 2a + c \times -2b + c \times 3c.$

$b \times 3c.$

$\langle -1, -1, 1 \rangle + \langle 6, 3, 6 \rangle + \langle -2, -2, 2 \rangle + \langle 3, -3, 3 \rangle$

$+ \langle -4, -2, -4 \rangle + \langle -1, 1, -1 \rangle$

~~$\langle 3, 3, 12 \rangle$~~

$\langle 3, -6, 9 \rangle$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0)$$ $\langle 0\ 1, -1 \rangle$

---

ans. The angle at A is: radians ; $\frac{\pi}{3}$

The angle at B is: radians ; $\frac{\pi}{3}$

The angle at C is: radians ; $\frac{\pi}{3}$

$\vec{AB} = (1,0,1) - (0,0,0) = \langle 1,0,1 \rangle$ ✓, $\vec{BA} = \langle -1,0,-1 \rangle$.

$\vec{BC} = \langle 1,1,0 \rangle - \langle 1,0,1 \rangle = \langle 0,1,-1 \rangle$ $\vec{CB} = \langle 0,-1,1 \rangle$

$\vec{AC} = \langle 1,1,0 \rangle$ $\vec{AC} = \langle -1,-1,0 \rangle$

Angle B: $\cos\theta = \dfrac{\langle 1,0,1 \rangle \cdot \langle 0,-1,+1 \rangle}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{0+0+1}{2} = +\dfrac{1}{2} = \cos\theta.$

Angle A: $\cos\theta = \dfrac{\langle 1,0,1 \rangle \cdot \langle 1,1,0 \rangle}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{1+0+0}{2} = \dfrac{1}{2} = \cos\theta$

Angle C: $\dfrac{\langle 0,-1,+1 \rangle \cdot \langle -1,-1,0 \rangle}{\sqrt{2} \cdot \sqrt{2}} = \dfrac{1}{2} = \cos\theta.$ $\theta = \dfrac{\pi}{3}.$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

---

ans. $-4\sqrt{3}$

$\nabla f = \langle 3x^2 + yz,\ 3y^2 + xz,\ 3z^2 + xy \rangle.$

$(-1,-1,-1) - (1,1,1) = \langle -2,-2,-2 \rangle$

$\| \langle -2,-2,-2 \rangle = \sqrt{2^2+2^2+2^2} = \sqrt{3 \cdot 4} = 2\sqrt{3}$

$u \ \langle -\frac{1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}} \rangle.$

$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle.$

$\langle 4,4,4 \rangle \cdot \langle -\frac{1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}},\ \frac{-1}{\sqrt{3}} \rangle.$

$= -\frac{4}{\sqrt{3}} \cdot 3 = \frac{-12}{\sqrt{3}} = \frac{-12\sqrt{3}}{3} = -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

$x = \cos(1)$ $y = \sin(1)$

ans. $6\cos^2(1) - 6\sin^2(1)$

$$\frac{dg}{du} = \frac{dg}{dx} \cdot \frac{dx}{du} + \frac{dg}{dy} \cdot \frac{dy}{du}$$

$\nabla g = \langle 6x, -6y \rangle$

$\nabla \cancel{x}_u = e^u \cos(v)$

$y_u = e^u \sin(v)$

$6x(e^u \cos v) - 6y(e^u \sin v)$

$6\cos(1)\cos(1)) - 6\sin(1)\sin(1)$

$6\cos^2(1) - 6\sin^2(1)$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3x + \cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x) \rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. $\frac{24\pi}{3} = 8\pi$

$d S 2$

$0 \le \theta \le \frac{\pi}{2}$

$\operatorname{div}(F)\ \langle 3, -2, 5 \rangle$

$=$

$\iint F \cdot dS = \iiint_D \operatorname{div}(F)\, dV$

$\iiint_D 6\, dV$

$6 \cdot \frac{4}{3} \pi 2^3$

$\frac{24 \cdot 8\pi}{3} \div 8 = \frac{24\pi}{3}$

$\frac{8 \cdot 4 \cdot 6}{3 \cdot 8} = \frac{24\pi}{3}$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. $-15.$

$\iint_S F \cdot dS$

$\int_0^1\int_0^1 -P\left(\frac{dg}{dx}\right) - Q\left(\frac{dg}{dy}\right) + R$

$g = 2x + 3y$

$g_x = 2$

$g_y = 3$

$\int_0^1\int_0^1 -3z(2) - 2x(3) + y + z \, dx\,dy$

$\int_0^1\int_0^1 -6z - 6x + y + z \, dy\,dx$

$\int_0^1\int_0^1 -6(2x+3y) - 6x + y + 2x + 3y \, dy\,dx$

$\int_0^1\int_0^1 -16x - 14y \, dy\,dx$

$\int_0^1 -16x - 7 \, dx$

$= -15$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. $\left(\frac{3}{4}, -1\right)$ is a saddle point.

$f_x = 4 - \frac{2}{2x+y}$

$f_y = -2y - \frac{1}{2x+y}$

$4 - \frac{2}{2x+y} = 0$

$-2y - \frac{1}{2x+y} = 0$

$-2y = \frac{1}{2x+y}$

$-4xy - 2y^2 = 1$

$4 = \frac{2}{2x+y}$

$8x + 4y = 2$

$y = \frac{2-8x}{4}$

Crit Point: $\left(\frac{3}{4}, -1\right)$

$f_{xx} = \frac{4}{(2x+y)^2} = 16 \qquad f_{yy} = \frac{1}{(2x+y)^2} - 2$

$= 2$

$f_{xy} = \frac{2}{(2x+y)^2} = 8$

$4 - \frac{2}{2x+y} = 0$

$4(2x+y) = 2$

$2x+y = \frac{1}{2}$

$y = \frac{1}{2} - 2x.$

Discriminant:

$16(-2) - 64$

$= -32$

$\rightarrow \left(\frac{3}{4}, -1\right)$ is a saddle point because discriminant < 0.

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$\frac{1}{2\sqrt{2x^2+3y^2+z^2}} \cdot 2x^2 \cdot 2x$$

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \ .$$

$$f(1,1,2) = \sqrt{2+3+4} = 3$$

---

ans. about 3.0003

$$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} \qquad f_x(1,1,2) = \frac{2}{\sqrt{2+3+4}} = \frac{2}{3}$$

$$f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} \qquad f_y(1,1,2) = 1$$

$$f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}} \qquad f_z(1,1,2) = \frac{2}{3}$$

$$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$$

$$L(1.001, 0.999, 2.001) = 3.0003$$

$$3 + \frac{2}{3}(1.001-1) + 1(0.999-1) + \frac{2}{3}(2.001-2)$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$y = x$ $r\cos\theta$ $r\sin\theta = r\cos\theta$ $1 = \tan\theta$

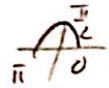
$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx\ .$$

$r = 1$ $x = r\cos\theta$ $y = r\sin\theta$

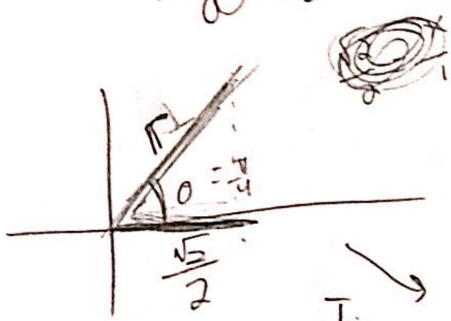
Explain! $r = 1$

---

ans.



$$\int_0^{\frac{\pi}{4}}\int_0^{\cos\theta} \cos\theta\, r\, dr\, d\theta + \int_{\frac{\pi}{4}}^{2\pi}\int_0^{\pi} \cos\theta\, r\, dr\, d\theta$$



$\cos\theta = \frac{\sqrt{2}}{2}$

$\cos\theta = \dfrac{\frac{\sqrt{2}}{2}}{r}$

Trig identity

$r = \dfrac{\frac{\sqrt{2}}{2}}{\cos\theta}$

$$\int_0^{\frac{\pi}{4}}\int_0^{\frac{\sqrt{2}}{2}\sec\theta} r\cos\theta \cdot r \cdot dr\, d\theta + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{1} r\sin\theta\, r\, dr\, d\theta.$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

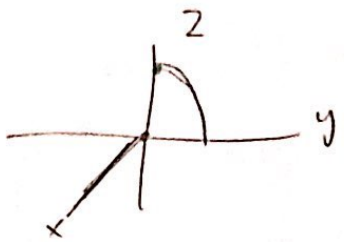
to **spherical coordinates. Do not evaluate.**

ans. $\int_0^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi)(\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\,d\theta\, d\phi\, d\rho.$

We can tell r=2 here

$x^2+y^2+z^2=2^2$

$\rho<2$



$$\int_0^2 \int_0^{\frac{\pi}{2}} \int_{\frac{\pi}{2}}^{\pi} (\rho\sin\phi\cos\theta)^2(\rho\sin\phi)(\sin\theta)(\rho\cos\phi)(\rho^2\sin\phi)\,d\theta\, d\phi\, d\rho.$$

$dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos(t), -3\sin(t)\rangle$

$r''(t) = \langle 0, -3\sin(t), -3\cos(t)\rangle$

$$\begin{pmatrix} i & j & k \\ 0 & 3(\frac{1}{2}) & -3(\frac{\sqrt{3}}{2}) \\ 0 & -3(\frac{\sqrt{3}}{2}) & -3\cdot\frac{1}{2} \end{pmatrix}$$

$r'(t) \times r''(t)$ when $t = \frac{\pi}{3}$

$= \langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle \times \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\rangle$

$= \langle -9, 0, 0\rangle$

$\|r'(t)\|^3 = \sqrt{\left(0^2 + \frac{9}{4} + \frac{27}{4}\right)}^3 = 27$

$\|r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3})\| = 9$

$\frac{9}{27}$

$\frac{1}{3}$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \,, uv \,, v^2 \rangle \quad , \quad 0 < u < v < 1 \ .$$

---

ans.

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16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14.

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle$

$\nabla f(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla g(1,1,1) \langle 1, 2, 3 \rangle$

$\langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle$

$1 + 4 + 9$

$= 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

$\frac{x^2-y^2}{(x-y)}$ $(x+y)$

ans. 0

let $x+y=a$

$z+w=b$

$\frac{\cancel{(a+b)(a-b)}}{\cancel{a-b}}$

$$\frac{a^2-b^2}{a-b} = a+b$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} x+y = 0+0 = 0.$$