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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$
3. $z = 0 + -3\sqrt{3}(x-\frac{\pi}{3}) + -5\sqrt{3}(y-\frac{\pi}{3}) + -6\sqrt{3}(z-\pi/3)$
4. $\langle -1, -6, 6 \rangle$
5. $\pi/3, \pi/3, \pi/3$
6. $-4\sqrt{3} = -12/\sqrt{3}$
7. $6\cos^2(1) - 6\sin^2(1)$
8. 8π
9. -15
10. Saddle point @ $(3/4, -1)$
11. $3.000\overline{3}$
12. $1/3$
13. $\int_0^2\int_0^{\pi}\int_{\pi/2}^{\pi} (\rho\cos\theta\sin\phi)^2(\rho\sin\theta\sin\phi)(\rho\cos\phi)\cdot\rho^2\sin\phi\, d\phi\, d\theta\, d\rho$
14. $1/3$
15. $\int_0^1\int_0^1 \sqrt{4v^4+16u^2v^2+4u^{4}}\,du\,dv$
16. 14
17. 0

Sign the following declaration:

I   Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Joseph Barr

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

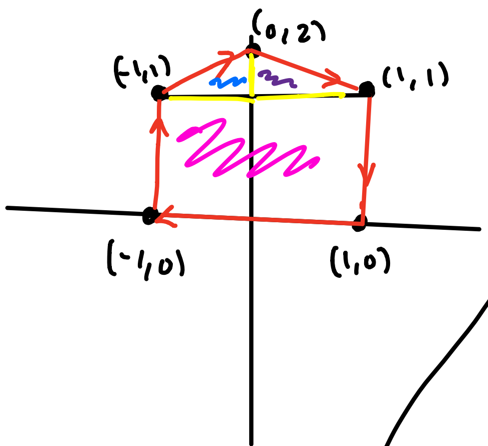
over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans. -18



$$-\iint_R (11-5)\,dA$$

* Using Greenes Theorem

$$-6 \cdot \left(\tfrac{1}{2}(1) + \tfrac{1}{2}(1) + 2(1) \right) = -18$$

$-\iint \text{curl}_z\, dA$

constant

The answer is negative because its going clockwise from starting point. Then after we get the constant 6, we find the area of the pentagon using geometry.

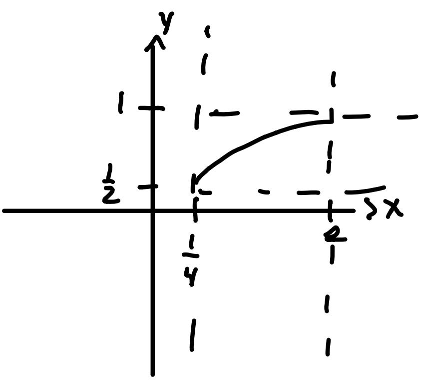
2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,dx\,dy \qquad .$$

---

ans. $\displaystyle\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$

$D = \{(x,y) \mid 0 \le y \le \sqrt{x},\ \frac{1}{4} \le x \le 1\}$



$D = \{(x,y) \mid \frac{1}{2} \le y \le 1,\ y^2 \le x \le 1\}$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \;=\; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = 0 + -3\sqrt{3}(x - \frac{\pi}{3}) + -5\sqrt{3}(y - \frac{\pi}{3}) + -6\sqrt{3}(z - \pi/3)$

$2\cos(\frac{\pi}{3}) + 4\cos(\frac{\pi}{3}) + 8\cos(\frac{\pi}{3}) = 7$

$1 + 2 + 4 = 7$

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x = -2\sin(x+y) - 4\sin(x+z) \rightarrow -2\sin(\frac{\pi}{3}) - 4\sin(\pi/3)$

$f_y = -2\sin(x+y) - 8\sin(y+z) \rightarrow -2\sin(\pi/3) - 8\sin(\pi/3)$

$f_z = -4\sin(x+z) - 8\sin(y+z) \rightarrow -4\sin(\pi/3) - 8\sin(\pi/3)$

$z = 0 + -3\sqrt{3}(x - \frac{\pi}{3}) + -5\sqrt{3}(y - \frac{\pi}{3}) + -6\sqrt{3}(z - \pi/3)$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $\langle -1, -6, 6 \rangle$

$$\cancel{(a \times 2a)} - (a \times b) + (a \times c) + (b \times 2a) - (b \times b) + (b \times 3c) + (c \times 2a) - (c \times b) + (c \times c) \cdot 3$$

$$0 - \langle 1, 1, -1 \rangle + \langle 2, 1, 2 \rangle + 2\langle -1, -1, 1 \rangle - 0 + 3\langle 1, -1, 1 \rangle + 2\langle -2, -1, -2 \rangle - \langle -1, 1, -1 \rangle$$

$\Sigma i = -1 + 2 - 2 + 3 - 4 + 1 = -1$

$\Sigma j = 1 + 1 - 2 - 3 - 2 - 1 = -6$

$\Sigma k = -1 + 2 + 2 + 3 - 4 + 1 = 6$

$\langle -1, -6, 6 \rangle$

5. (12 points) Find the three angles of the triangle ABC where

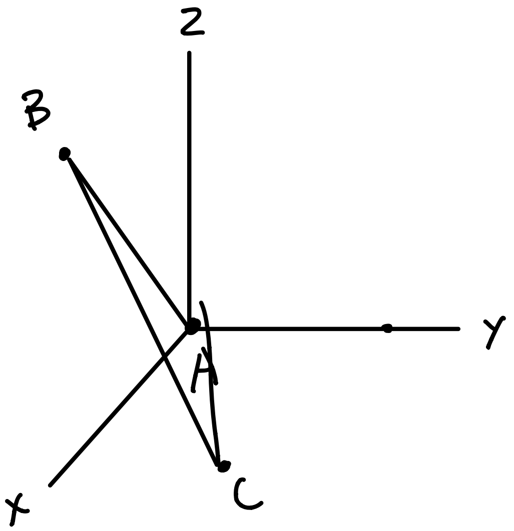
$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \qquad .$$

---

ans. The angle at A is: radians ; $\pi/3$

The angle at B is: radians ; $\pi/3$

The angle at C is: radians ; $\pi/3$



$r_{AB} = \langle 1,0,1 \rangle$

$r_{AC} = \langle 1,1,0 \rangle$

$r_{BC} = \langle 0,1,-1 \rangle$

$\langle 0,-1,1 \rangle \cdot \langle -1,-1,0 \rangle$

r

$\theta_A = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta_A = \pi/3$

$\theta_B = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta_C = \cos^{-1}\left(\frac{1}{2}\right)$

$\vec{r}_{AB} \cdot \vec{r}_{AC} = \|r_{AB}\| \|r_{AC}\| \cos\theta_A$

$\theta_A = \cos^{-1}\left(\frac{\vec{r}_{AB} \cdot \vec{r}_{AC}}{\|r_{AB}\| \|r_{AC}\|}\right)$

$\vec{r}_{CB} \cdot \vec{r}_{CA} = \|\vec{r}_{BC}\| \cdot \|\vec{r}_{CA}\| \cos\theta_C$

$\theta_C = \cos^{-1}\left(\frac{\vec{r}_{CB} \cdot \vec{r}_{CA}}{\|\vec{r}_{CB}\| \cdot \|\vec{r}_{CA}\|}\right)$

$\vec{r}_{BA} \cdot \vec{r}_{BC} = \|\vec{r}_{BA}\| \|\vec{r}_{BC}\| \cos\theta_B$

$\theta_B = \cos^{-1}\left(\frac{\vec{r}_{BA} \cdot \vec{r}_{BC}}{\|r_{BA} \cdot r_{BC}\|}\right)$

* Its an equilateral triangle

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $\overset{P}{(1,1,1)}$ in a direction pointing to the point $\overset{Q}{(-1,-1,-1)}$.

---

ans. $-4\sqrt{3}$

$\nabla f = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle \Rightarrow \nabla f_P = \langle 4, 4, 4 \rangle$

$\vec{QP} = \langle -1-1, -1-1, -1-1 \rangle = \langle -2, -2, -2 \rangle$

$\vec{e}_{QP} = \left\langle \frac{-2}{\sqrt{2^2+2^2+2^2}}, \frac{-2}{\sqrt{2^2+2^2+2^2}}, \frac{-2}{\sqrt{2^2+2^2+2^2}} \right\rangle = \left\langle -\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}} \right\rangle$

$\rightarrow \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

$\nabla f_P \cdot \vec{e}_{QP} = \langle 4, 4, 4 \rangle \langle -1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3} \rangle = -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} = \frac{-12}{\sqrt{3}}$

$D = \nabla f_P \cdot \vec{e}_{QP} = \frac{-12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-12\sqrt{3}}{3} = -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos^2(1) - 6\sin^2(1)$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u} \; ; \quad \frac{\partial g}{\partial x} = 6x \quad \frac{\partial g}{\delta y} = -6y$$

$$\frac{\partial x}{\partial u} = e^u \cos v \quad \frac{\delta y}{\delta u} = e^u \sin v$$

$$\frac{\partial g}{\partial u} = 6e^u \cos v \cdot e^u \cos(v) - 6e^u \sin(v) \cdot e^u \sin(v)$$

$$= 6e^{2u}\cos^2(v) - 6e^{2u}\sin^2(v)$$

$$\left.\frac{\partial g}{\delta u}\right|_{(0,1)} = 6\cos^2(1) - 6\sin^2(1)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

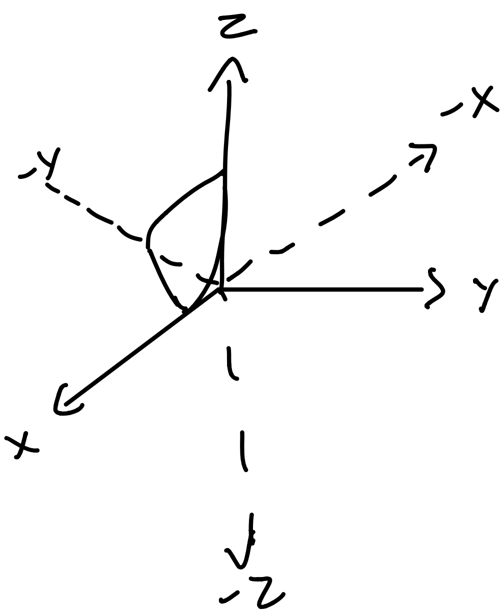
---

ans. 8π

$\mathrm{div}(F) = 6$ Divergence Theorem

$6 \iiint_B dV = 6 \cdot$ Volume of part of sphere

$6\left(\frac{4}{3}\pi 2^3\right)\frac{1}{8} = 8\pi$



$$\int\int_S \mathbf{F}\cdot d\mathbf{S} =$$
$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad .$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15

$G(u,v) = \langle u, v, 2u+3v \rangle \Rightarrow F(G(u,v)) = \langle 3(2u+3v), 2u, v+2u+3v \rangle$
$= \langle 6u+9v, 2u, 2u+4v \rangle$

$G_u = \langle 1, 0, 2 \rangle$

$G_v = \langle 0, 1, 3 \rangle$

$N = G_u \times G_v = \langle -2, -3, 1 \rangle$

$\int_0^1 \int_0^1 F(G(u,v)) \cdot N \, du\,dv = \int_0^1 \int_0^1 \langle 6u+9v, 2u, 2u+4v \rangle \cdot \langle -2, -3, 1 \rangle \, du\,dv$

$\int_0^1 \int_0^1 \left(-2(6u+9v) - 3(2u) + 2u+4v\right) du\,dv = \int_0^1 \int_0^1 (-16u - 14v)\, du\,dv$

$-12u - 18v - 6u + 2u + 4v$

$-18u - 18v + 2u + 4v$

$-16u - 14v$

① $\int_0^1 (-16u - 14v)\, du = -8u^2 - 14uv \Big|_0^1 = -8 - 14v \rightarrow$ ② $\int_0^1 -8 - 14v \, dv =$

$-8v - 7v^2 \Big|_0^1$

$\boxed{-8 - 7 = -15}$

$4 - \frac{2}{\frac{3}{4}}$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. Saddle point @ $(3/4, -1)$

$\hat{f}_x = 4 - \frac{2}{2x+y}$;

$\hat{f}_{xx} = \frac{4}{(2x+y)^2}$

$\hat{f}_y = -2y - \frac{1}{2x+y}$

$\hat{f}_{yy} = \frac{1}{(2x+y)^2} - 2$

$f_{xy} = \frac{2}{(2x+y)^2}$

$f_x = 0 \Rightarrow 4 - \frac{2}{2x+y} = 0$

$f_y = 0 \Rightarrow -2y - \frac{1}{2x+y} = 0$

$\boxed{\text{cps: } (\tfrac{3}{4}, -1) \quad P}$

$2(2x+y) = 1$

$2x+y = \frac{1}{2}$

$y = \frac{1}{2} - 2x$

$-2(\frac{1}{2} - 2x) - \frac{1}{\cancel{2x} + \frac{1}{2} - \cancel{2x}} = 0$

$-2(\frac{1}{2} - 2x) - 2 = 0$

$\frac{1}{2} - 2x = -1$

$-2x = -\frac{3}{2}$

$x = \frac{3}{4} \quad y = \frac{1}{2} - 2(\frac{3}{4}) = \frac{1}{2} - \frac{3}{2} = -1$

$D = f_{xx}(P) \cdot f_{yy}(P) - f_{xy}(P)$

$= \frac{4}{(2(3/4) + -1)^2} \cdot \left(-2(-1) - \frac{1}{2(3/4) + -1}\right) - \frac{2}{(2(\frac{3}{4}) + -1)^2} = -8$

$\boxed{\text{Saddle point @ } (3/4, -1)}$

$D = -8 < 0$ which means its a saddle point, accordin to the Second deravite test.

Critical point: $(3/4, -1)$ and is saddle point

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) \;=\; \sqrt{2x^2+3y^2+z^2} \quad .$$

---

ans. $3.000\overline{3}$

$P = (1,1,2)$

$\hat{f}_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x \Rightarrow \frac{2}{\sqrt{2+3+2^2}} = \frac{2}{\sqrt{2+3+4}} = \frac{2}{\sqrt{9}} = \frac{2}{3}$

$\hat{f}_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y \Rightarrow \frac{3}{\sqrt{2+3+2^2}} = 1$

$\hat{f}_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 2z \Rightarrow \frac{2}{\sqrt{2+3+2^2}} = \frac{2}{3}$

$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$

$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(0.001) + (-.001) + \frac{2}{3}(0.001)$

$= 3.000\overline{3}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx.$$

Explain!

---

ans. $\frac{1}{3}$

$D_1 = \{(x,y) \mid 0 \le y \le x,\ 0 \le x \le \frac{\sqrt{2}}{2}\} \Rightarrow D = \{(r,\theta) \mid 0 \le r \le 1,\ \frac{\pi}{4} \le \theta \le \frac{\pi}{2}\}$

$y = x$

$r\sin\theta = r\cos\theta$

$\tan\theta = 1$

$\theta = \frac{\pi}{4}$

$r\cos\theta = \frac{\sqrt{2}}{2} \Rightarrow \theta = \frac{\pi}{4}$

$r\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$

*assuming we are integrating over unit circle

1 $\int_0^1 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cos\theta\,d\theta\,dr \Rightarrow \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} r^2\cos\theta\,d\theta \Rightarrow$

$\Rightarrow r^2 \cdot (\sin\theta)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \Rightarrow \int_0^1 r^2\left(1 - \frac{\sqrt{2}}{2}\right)dr \rightarrow$

$\rightarrow \frac{r^3}{3}\Big|_0^1 \cdot \left(1 - \frac{\sqrt{2}}{2}\right) = \frac{1}{3} - \frac{\sqrt{2}}{6}$

$D = \{(x,y) \mid 0 \le y \le \sqrt{1-x^2},\ \frac{\sqrt{2}}{2} \le x \le 1\} \Rightarrow \{(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \frac{\pi}{4}\}$

$y = \sqrt{1-x^2}$

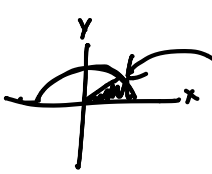
$y^2 + x^2 = 1$

$r = 1$

$r\cos\theta = 1$

$r\cos\theta = \frac{\sqrt{2}}{2}$

$\theta = [0, \frac{\pi}{4}]$



Semicircle (part)

2 $\int_0^1 \int_0^{\pi/4} r^2\cos\theta\,d\theta\,dr \rightarrow$ 1 $\int_0^{\pi/4} r^2\cos\theta\,d\theta =$

$r^2 \cdot \left(\sin\theta\Big|_0^{\pi/4}\right) = r^2 \cdot \frac{\sqrt{2}}{2} \Rightarrow \int_0^1 r^2 \cdot \frac{\sqrt{2}}{2}\,dr =$

$\frac{\sqrt{2}}{6}$

$\frac{1}{3} - \frac{\sqrt{2}}{6} + \frac{\sqrt{2}}{6} = \frac{1}{3}$

; I used the substitution $x = r\cos\theta$ and $y = \sin\theta \cdot r$ to find the bounds and sketched the domain.

13. (12 points) Convert the triple iterated integral

$$\int_0^{\cancel{4}\;2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\displaystyle\int_0^2 \int_0^{\pi} \int_{\pi/2}^{\pi} (\rho\cos\theta\sin\phi)^2 (\rho\sin\theta\sin\phi)(\rho\cos\phi)\cdot \rho^2\sin\phi \, d\phi\, d\theta\, d\rho$

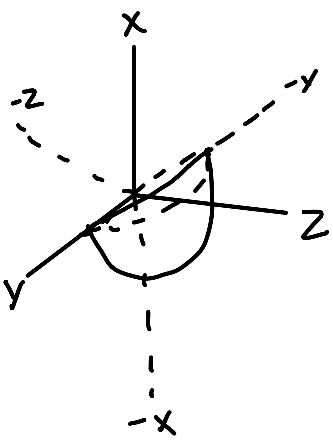
$D = \{(x,y,z) \mid -\sqrt{4-z^2-y^2} \le x \le 0,\ 0 \le y \le \sqrt{4-z^2},\ 0 \le z \le 2\}$

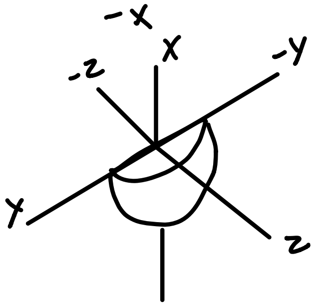
$= \{\rho, \theta, \phi \mid 0 \le \rho \le 2,\ 0 \le \theta \le \pi,\ \frac{\pi}{2} \le \phi \le \pi\}$

$x^2 + y^2 + z^2 = 4$

$y^2 + z^2 = 4$

$z = 0\ ;\ z = 2$





14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $1/3$

$$K(t) = \frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$r'(t) = \langle 0, 3\cos(t), -3\sin(t) \rangle$

$r''(t) = \langle 0, -3\sin(t), -3\cos(t) \rangle$

$r'(\frac{\pi}{3}) = \langle 0, \frac{3}{2}, \frac{-3\sqrt{3}}{2} \rangle$

$r''(\frac{\pi}{3}) = \langle 0, \frac{-3\sqrt{3}}{2}, -\frac{3}{2} \rangle$

$$\frac{\|\langle -9, 0, 0\rangle\|}{\|\langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle\|^3} = \frac{9}{3^3} = \frac{1}{3}$$

$$\left(\frac{3}{2}\right)^2 + \left(\frac{3\sqrt{3}}{2}\right)^2$$

$$\frac{9}{4} + \frac{27}{4} = \frac{36}{4} = 9$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) \;=\; \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\displaystyle\int_0^1\int_0^1 \sqrt{4v^4+16u^2v^2+4u^{2}}\;du\,dv$

$$\int_0^1\int_0^1 \sqrt{4v^4+16u^2v^2+4u^{2}}\;du\,dv$$

$r_u = \langle 2u, v, 0\rangle$

$r_v = \langle 0, u, 2v\rangle$

$$N = r_u \times r_v = \begin{bmatrix} 2u \\ v \\ 0 \end{bmatrix} \times \begin{bmatrix} 0 \\ u \\ 2v \end{bmatrix} = \begin{bmatrix} 2v^2 - 0 \\ -4uv \\ 2u^2 - 0 \end{bmatrix} = \langle 2v^2, -4uv, 2u^2\rangle$$

$$\|N\| = \sqrt{4v^4+16u^2v^2+4u^2}$$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle \Rightarrow \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle \Rightarrow \langle 1, 2, 3 \rangle$

$\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1 + 2^2 + 3^2 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. $L = 0$

$\lim_{x \to 0} \frac{x^2}{x} = 0$

$\lim_{y \to 0} \frac{y^2}{y} = 0$

$\lim_{z \to 0} \frac{-z^2}{-z} = 0$

$\lim_{w \to 0} \frac{-w^2}{w} = 0$

The limit exists. $L = 0$
because the
limit along the
axes are equal