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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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Sign the following declaration:

I Jinquan Lin Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Jinquan Lin

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. -18

2. $\int_{\frac{1}{2}}^{1}\int_{y^2}^{1} f(x,y)\,dxdy+\int_{0}^{\frac{1}{2}}\int_{\frac{1}{4}}^{1} f(x,y)\,dxdy$

3. $z=-\frac{x}{2}-\frac{5}{6}y+\frac{7}{18}\pi$

4. $3i-6j+9k$

5. $A=\frac{\pi}{3}$, $B=\frac{\pi}{3}$, $C=\frac{\pi}{3}$

6. $-4\sqrt{3}$

7. $6\cos 2$

8. 0

9. $15\sqrt{14}$

10. $(\frac{3}{4},-1)$ is local minimum

11. $\frac{9.001}{3}$

12. $\frac{\sqrt{2}}{6}$

13. $\int_{0}^{2}\int_{\frac{\pi}{2}}^{\pi}\int_{\frac{\pi}{2}}^{\pi} \rho^2\sin^2\theta\cos^2\varphi\cdot\rho\sin\theta\sin\varphi\cdot\rho\cos\theta\cdot\rho^2\sin\theta\, d\rho\, d\theta\, d\varphi$

14. $\frac{1}{3}$

15. $\int_{0}^{1}\int_{0}^{1} 2\sqrt{v^4+u^4+4u^2v^2}\,dv\,du$

16. 14

17. the limit exist at 0.

1. $\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy$

$P = \cos(e^{\sin x}) + 5y$, $Q = \sin(e^{\cos y}) + 11x$

$\frac{dP}{dy} = 5$, $\frac{dQ}{dx} = 11$

Using Green formula

$\iint_D \left(\frac{dQ}{dx} - \frac{dP}{dy}\right) ds$

$= \oint P\,dx + Q\,dy$

the path given is clockwise

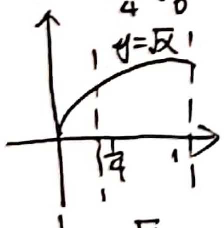
$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy$

$= -\iint_D 6\,dS = -6S$

$= -6 \times \frac{(2+1)\times 1}{2} \times 2$

$= -18$

2. $\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dx\,dy$



$\begin{cases} y^2 \le x \le 1 \\ \frac{1}{2} \le y \le 1 \end{cases}$ and $\begin{cases} \frac{1}{4} \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dx\,dy = \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy + \int_{0}^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy$

3. $2\cos(x+y)+4\cos(x+z)+8\cos(y+z)=7$

$F(x,y,z)=2\cos(x+y)+4\cos(x+z)+8\cos(y+z)-7$

$F_x=-2\sin(x+y)-4\sin(x+z)$

$F_y=-2\sin(x+y)-8\sin(y+z)$

$F_z=-4\sin(x+z)-8\sin(y+z)$

At point $(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6})$

$\vec{n}=\langle F_x, F_y, F_z\rangle=(-3\sqrt{3},-5\sqrt{3},-6\sqrt{3})$

$3(x-\frac{\pi}{6})+5(y-\frac{\pi}{6})+6(z-\frac{\pi}{6})=0$

$z=-\frac{x}{2}-\frac{5}{6}y+\frac{7}{18}\pi$

4. $(a+b+c)\times(2a-b+3c)$

$=a\times 2a-a\times b+3a\times c+2b\times a-b\times b+3b\times c+2c\times a-c\times b+3c\times c$

$=-3a\times b+a\times c+4b\times c$

$=-3(i+j-k)+(2i+j+2k)+4(i-j+k)$

$=3i-6j+9k$

5. $A=(0,0,0)$, $B=(1,0,1)$, $C=(1,1,0)$

$\vec{AB}=(1,0,1)$, $\vec{AC}=(1,1,0)$

$\cos A=\frac{AB\cdot AC}{|AB||AC|}=\frac{1}{\sqrt{2}\cdot\sqrt{2}}=\frac{1}{2}$

$A=\cos^{-1}(\frac{1}{2})=\frac{\pi}{3}$

$BA=(-1,0,-1)$, $BC=(0,1,-1)$

$\cos B=\frac{BA\cdot BC}{|BA||BC|}=\frac{1}{2}$

$B=\cos^{-1}(\frac{1}{2})=\frac{\pi}{3}$

$C=\pi-\frac{\pi}{3}-\frac{\pi}{3}=\frac{\pi}{3}$

6. $f(x,y,z) = x^3 + y^3 + z^3 + xyz$

$\text{grad} = (3x^2+yz)i + (3y^2+xz)j + (3z^2+xy)k$

At $(1,1,1)$ $\vec{n} = \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

$$\frac{df}{dT} = (4i+4j+4k)\cdot\left(-\frac{i}{\sqrt{3}} - \frac{j}{\sqrt{3}} - \frac{k}{\sqrt{3}}\right)$$

$$= -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}}$$

$$= -4\sqrt{3}$$

7. $g(x,y) = 3x^2 - 3y^2$, $x = e^u \cos v$, $y = e^u \sin v$

$$\frac{dg}{du} = \frac{dg}{dx}\cdot\frac{dx}{du} + \frac{dg}{dy}\cdot\frac{dy}{du}$$

$$= 6x\cos v \cdot e^u + (-6y)\sin v e^u$$

$(u,v) = (0,1)$

$x = \cos 1$, $y = \sin 1$

$$\frac{dg}{du} = 6\cos 1\cdot\cos 1\, e^0 - 6\sin 1 \sin 1\, e^0$$

$$= 6(\cos^2 1 - \sin^2 1)$$

$$= 6\cos 2$$

8. $F = \langle 3x + \cos(y^3+yz),\ -2y + e^{x+z^2},\ 5z + \sin(xy^3+e^x)\rangle$

S is closed surface $\oiint_S \vec{F}\,\vec{ds} = \Delta\Phi = 0$

$\iint_S \vec{F}\,\vec{ds}$

$$= \iint_{D_{yz}} \vec{F}\cdot\vec{i}\,dydz + \iint_{D_{xy}} \vec{F}\cdot k\,dxdy - \iint_{D_{xz}} \vec{F}\cdot j\,dxdz + \iint_S \vec{F}\cdot\vec{N}\,dA$$

$$= (F_x - F_y + F_z)\pi + \iint (F_x x + F_y y + F_z z)\,dA$$

$$= (F_x - F_y + F_z)\pi + \iint (F_x r\sin\theta\cos\varphi + F_y r\sin\theta\sin\varphi + F_z\cdot r\cos\theta)\, r^2\, d\theta\, d\varphi$$

$= 0$

9. $F = \langle 3z, 2x, y+z \rangle$

$F = 2x + 3y - z \Rightarrow \vec{N} = \left(\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}\right) = (2, 3, -1)$

$$\iint_S \vec{F}\, d\vec{s} = \iint_S \vec{F} \cdot \vec{N}\, dA$$

$$= \iint_S (6x - y + 5z)\, dA$$

$$= \iint_{Dxy} (6x - y + 10x + 5y) \cdot \sqrt{1 + \left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2}\, dx\, dy$$

$$= \iint_{Dxy} \sqrt{14}(16x + 14y)\, dx\, dy$$

$$= 15\sqrt{14}$$

10. $f(x,y) = 4x - y^2 - \ln(2x+y)$

$$\frac{df}{dx} = 4 - \frac{2}{2x+y} = \frac{2(4x+2y-1)}{2x+y}$$

$$\frac{df}{dy} = -2y - \frac{1}{2x+y} = -\frac{2y^2+4xy+1}{2x+y}$$

$$\begin{cases} \frac{df}{dx} = 0 \\ \frac{df}{dy} = 0 \end{cases} \Rightarrow \begin{cases} 4x+2y-1=0 \\ 2y^2+4xy+1=0 \end{cases} \Rightarrow \begin{cases} x = \frac{3}{4} \\ y = -1 \end{cases}$$

$$\begin{cases} (\frac{df}{dx})^2 = \frac{4}{(2x+y)^2} \\ (\frac{df}{dy})^2 = -2 - \frac{1}{(2x+y)^2} \\ \frac{df}{dxdy} = \frac{2}{(2x+y)^2} \end{cases} \Rightarrow \begin{cases} (\frac{df}{dx})^2 = 16 \\ (\frac{df}{dy})^2 = -6 \\ \frac{df}{dxdy} = 8 \end{cases}$$

~~$16 \times (-6) - 8^2 \Leftrightarrow 0$~~

$16 \times 8 - (-6)^2 > 0$ and $16 > 0$

$(\frac{3}{4}, -1)$ is a local minimum

11. $f(x,y,z)=\sqrt{2x^2+3y^2+z^2}$

$$\frac{df}{dx}=\frac{1}{2}\cdot\frac{4x}{\sqrt{2x^2+3y^2+z^2}}=\frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$\frac{df}{dy}=\frac{1}{2}\frac{6y}{\sqrt{2x^2+3y^2+z^2}}=\frac{3y}{\sqrt{2x^2+3y^2+z^2}}$$

$$\frac{df}{dz}=\frac{1}{2}\frac{2z}{\sqrt{2x^2+3y^2+z^2}}=\frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

$$df=\frac{df}{dx}dx+\frac{df}{dy}dy+\frac{df}{dz}dz$$

$(x,y,z)=(1,1,2)$

$(dx,dy,dz)=(0.001,-0.001,0.001)$

$df=\frac{2}{3}\times 0.001-\frac{3}{3}\times 0.001+\frac{2}{3}\times 0.001$

$=\frac{1}{3}\times 0.001$

$f(x,y,z)=3$

$f(1.001,0.999,2.001)=3+\frac{1}{3}\times 0.001=\frac{9.001}{3}$

12. $\int_0^{\frac{\sqrt{2}}{2}}\int_0^{x} x\,dy\,dx+\int_{\frac{\sqrt{2}}{2}}^{2}\int_0^{\sqrt{1-x^2}} x\,dy\,dx$

$\begin{cases} 0\le r\le 1 \\ 0\le\theta\le\frac{\pi}{4}\end{cases}$ $x=r\cos\theta,\ y=r\sin\theta$

$=\int_0^1\int_0^{\frac{\pi}{4}} r\cos\theta\, r\,d\theta\,dr$

$=\int_0^1 r^2dr\cdot\int_0^{\frac{\pi}{4}}\cos\theta\,d\theta$

$=\frac{\sqrt{2}}{2}\cdot\frac{1}{3}$

$=\frac{\sqrt{2}}{6}$

13. $\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 y z \, dx\, dy\, dz$

$$\begin{cases} x = \rho \sin\theta \cos\varphi \\ y = \rho \sin\theta \sin\varphi \\ z = \rho \cos\theta \end{cases}$$

$$\text{Jacobian} = \begin{vmatrix} \frac{dx}{d\rho} & \frac{dx}{d\theta} & \frac{dx}{d\varphi} \\ \frac{dy}{d\rho} & \frac{dy}{d\theta} & \frac{dy}{d\varphi} \\ \frac{dz}{d\rho} & \frac{dz}{d\theta} & \frac{dz}{d\varphi} \end{vmatrix} = \sout{\rho^2 \sin\varphi}\ \rho^2 \sin\theta$$

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 y z \, dx\, dy\, dz$$

$$= \int_0^{2} \int_{\frac{\pi}{2}}^{\pi} \int_{\frac{\pi}{2}}^{\pi} \rho^2 \sin^2\theta \cos^2\varphi \cdot \rho \sin\theta \sin\varphi \cdot \rho \cos\theta \cdot \rho^2 \sin\theta \, d\rho\, d\theta\, d\varphi$$

~~$= \int_0^2 \rho^5 d\rho \int_{\frac{\pi}{2}}^{\pi} \sin^4\theta \cos\theta \, d\theta \cdot \int_{\frac{\pi}{2}}^{\pi} \sin\varphi \cos^2\varphi \, d\varphi$~~

~~$= \frac{32}{3} \times \left(-\frac{1}{4}\right) \times \frac{\pi}{16}$~~

~~$= -\frac{\pi}{6}$~~

~~14. $r(t) = \langle 5, 3\sin t, 3\cos t \rangle$~~

~~$\frac{dr}{dt} = \langle 0, 3\cos t, -3\sin t \rangle = \vec{v}$.~~

~~$\frac{d|r|}{dt} = 0$, $\frac{d|v|}{dt} = 0$.~~

~~$\frac{dr^2}{dt^2} = \langle 0, -3\sin t, -3\cos t \rangle = a$~~

~~$\left|\frac{dr^2}{dt^2}\right|$~~ $= \sqrt{0^2 + (3c}$

14. $r(t) = \langle 5, 3\sin t, 3\cos t \rangle$

$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$\|r'(t)\| = \sqrt{0^2 + (3\cos t)^2 + (-3\sin t)^2} = 3$

$\|r'(t)\|^3 = 27$

$r'(t) \times r''(t) = (-9, 0, 0)$

$\|r'(t) \times r''(t)\| = \sqrt{(9)^2 + 0^2 + 0^2} = 9$

$k(t) = \frac{1}{3} \Rightarrow k(\frac{\pi}{3}) = \frac{1}{3}$

15. $r(u,v) = \langle u^2, uv, v^2 \rangle$

$S = \int_S dS$

$= \iint_D \sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dy}{dz})^2}\, dx\, dz$

$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = v \cdot \frac{1}{2\sqrt{x}} = \frac{v}{2u}$

$\frac{dy}{dz} = \frac{dy}{dv} \cdot \frac{dv}{dz} = u \cdot \frac{1}{2\sqrt{z}} = \frac{u}{2v}$

$S = \iint_D \sqrt{1 + (\frac{v}{2u})^2 + (\frac{u}{2v})^2}\; 2u\,du \cdot 2v\,dv$

$=$ ~~$\int_0^1 \int_0^1 2(u^2+v^2)\,du\,dv$~~ $\int_0^1 \int_0^1 2\sqrt{v^4 + u^4 + 4v^2u^2}\, dv\, du$

16. $f(x,y,z) = xy^2z^3 \quad g(x,y,z) = x + y^2 + z^3$

$\text{grad}(f) = (y^2z^3, 2xyz^3, 3xy^2z^2)$

$\text{grad}(g) = (1, 2y, 3z^2)$

$\text{grad}(f) \cdot \text{grad}(g) = y^2z^3 + 4xy^2z^3 + 9xy^2z^4$

At $(1,1,1)$

$\text{grad}(f) \cdot \text{grad}(g) = 1 + 4 + 9$

$= 14$

17. $$\frac{(x+y)^2-(z+w)^2}{(x+y)-(z+w)} = x+y+z+w$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}$$

$$= \lim_{(x,y,z,w)\to(0,0,0,0)} (x+y+z+w)$$

$$= 0$$

The limit exist and equal zero.

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans.

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2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, dx\, dy \qquad .$$

---

ans.

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3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \;=\; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z =$

---

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans.

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5. (12 points) Find the three angles of the triangle ABC where

$$A = (0, 0, 0) \quad , \quad B = (1, 0, 1) \quad , \quad C = (1, 1, 0) \quad .$$

---

ans. The angle at A is: radians ;

The angle at B is: radians ;

The angle at C is: radians ;

---

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans.

---

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans.

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8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\, ,\, -2y + e^{x+z^2}\, ,\, 5z + \sin(xy^3 + e^x)\, \rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans.

---

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z \, , \, 2x \, , \, y + z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans.

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10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans.

---

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans.

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12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\, dy\, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\, dy\, dx \quad .$$

Explain!

---

ans.

---

13. (12 points) Convert the triple iterated integral

$$\int_0^4 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^0 x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans.

---

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans.

---

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans.

---

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans.

---

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans.
