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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

1.

2. $\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$

3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{\pi}{4}$

4. $-3j + 6K$

5.

6. $-4\sqrt{3}$

7. $6\left(\cos^2(1) - \sin^2(1)\right)$

8. 8π

9. -15

10. $(-\frac{1}{4}, 1)$ local min

11. $3.000\overline{3}$

12.

13. $\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{2} \rho^6 \sin^4\phi \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\theta\, d\phi$

14. $1/3$

15.

16. 14

17.

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y})\;+\;11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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ans.

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2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x, y)\, dy\, dx \quad .$$

127

448

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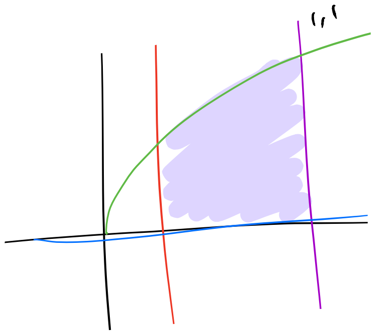
ans. $\int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$



$y = \sqrt{x}$ $y = 0$ $x = 1$ $x = \frac{1}{4}$

$x = y^2$

$\int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{\pi}{4}$

$f_x = -2\sin(x+y) - 4\sin(x+z)$

$f_x(\pi/6, \pi/6, \pi/6) = -\cancel{2}\left(\frac{\sqrt{3}}{\cancel{2}}\right) - 4\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} - 2\sqrt{3} = -3\sqrt{3}$

$f_y = -2\sin(x+y) - 8\sin(y+z)$

$f_y(\pi/6, \pi/6, \pi/6) = -2\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right) = -\sqrt{3} - 4\sqrt{3} = -5\sqrt{3}$

$f_z = -4\sin(x+z) - 8\sin(y+z)$

$f_z(\pi/6, \pi/6, \pi/6) = -4\left(\frac{\sqrt{3}}{2}\right) - 8\left(\frac{\sqrt{3}}{2}\right) = -2\sqrt{3} - 4\sqrt{3} = -6\sqrt{3}$

$-3\sqrt{3}(x - \pi/6) - 5\sqrt{3}(y - \pi/6) - 6\sqrt{3}(z - \pi/6) = 0$

$-3\sqrt{3}x + \frac{\pi\sqrt{3}}{2} - 5\sqrt{3}y + \frac{5\pi\sqrt{3}}{6} - 6\sqrt{3}z + \pi\sqrt{3}$

$-3\sqrt{3}x - 5\sqrt{3}y - 6\sqrt{3}z + \frac{3\pi\sqrt{3}}{2} = 0$

$\frac{\cancel{6\sqrt{3}}z}{\cancel{6\sqrt{3}}} = \frac{-3\cancel{\sqrt{3}}x - 5\cancel{\sqrt{3}}y + \frac{3}{2}\pi\cancel{\sqrt{3}}}{6\cancel{\sqrt{3}}}$

$= z = -\frac{1}{2}x - \frac{5}{6}y + \frac{\pi}{4}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $-3j + 6K$

$(a \times 2a) - (a \times b) + (a \times 3c) + (b \times 2a) - (b \times b) + (b \times 3c) + (c \times 2a) - (c \times b) + (c \times 3c)$

$\underset{0}{2(a \times a)} - (a \times b) + 3(a \times c) + 2(b \times a) - \underset{0}{(b \times b)} + \underset{0}{3(c \times c)} + 2(c \times a) - (c \times b) + \underset{0}{3(c \times c)}$

$-(a \times b) + 3(a \times c) + \underset{-(a \times b)}{2(b \times a)} + \underset{-(a \times c)}{2(c \times a)} - \underset{-(b \times c)}{(c \times b)}$

$-(a \times b) + 3(a \times c) - 2(a \times b) - 2(a \times c) + (b \times c)$

$-(i + j - K) + 3(2i + j + 2K) - 2(i + j - K) - 2(2i + j + 2K) + (i - j + K)$

$-i - j + K + 6i + 3j + 6K - 2i - 2j + 2K - 4i - 2j - 4K + i - j + K$

$\underset{0}{-i + 6i - 2i - 4i + i} \quad \underset{-3j}{-\ j + 3j - 2j - 2j - j} \quad \underset{+6K}{+\ K + 6K + 2K - 4K + K}$

$-3j + 6K$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: radians ;

The angle at B is: radians ;

The angle at C is: radians ;

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6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $\overset{P}{(1, 1, 1)}$ in a direction pointing to the point $\overset{Q}{(-1, -1, -1)}$.

---

ans. $-4\sqrt{3}$

$\vec{PQ} = \langle -2, -2, -2 \rangle$

$\| \vec{PQ} \| = \sqrt{4+4+4} = \sqrt{12} \longrightarrow \frac{1}{2\sqrt{3}} \langle -2, -2, -2 \rangle$

$\nabla f = \langle 3x^2 + yz,\ 3y^2 + xz,\ 3z^2 + xy \rangle$

$\nabla f(-1, -1, -1) = \langle 4, 4, 4 \rangle$

$\frac{1}{2\sqrt{3}} \langle -2, -2, -2 \rangle \cdot \langle 4, 4, 4 \rangle = \frac{1}{2\sqrt{3}} (-8 - 8 - 8)$

$= \frac{1}{2\sqrt{3}} (-24)$

$= -12/\sqrt{3}$

$= \frac{-12\sqrt{3}}{3} = -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

h g
u v   x y
r q   u v

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\left(\cos^2(1) - \sin^2(1)\right)$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$$

$$= 6x \cdot \cos v \cdot e^u + \left(-6y \cdot \sin v \cdot e^u\right)$$

at $(u,v) = (0,1)$:

$$= 6(0) - 6 \cdot \sin 1 \cdot e$$

$$-6e \sin(1)$$

$$x = e^0 \cdot \cos(1) = \cos(1)$$
$$y = e^0 \cdot \sin(1) = \sin(1)$$

$$6\cos(1) \cdot \cos(1) \cdot e^0 - 6\sin(1) \cdot \sin(1) \cdot e^0$$

$$= 6\cos^2(1) - 6\sin^2(1)$$

$$6\left(\cos^2 1 - \sin^2 1\right)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

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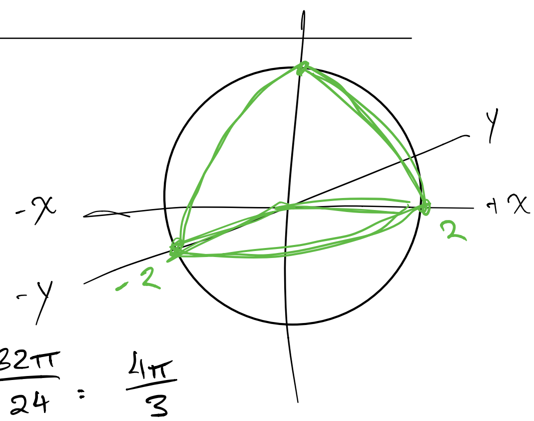
ans. 8π



div F : $3 + (-2) + 5 = 6$

Region is : $\frac{1}{8}$ sphere, rad 2 ,

$V = \frac{4}{3}\pi r^3 = \frac{4}{3}\cdot\pi\cdot 8 = \frac{32\pi}{3} / 8 = \frac{32\pi}{24} = \frac{4\pi}{3}$

$\frac{4\pi}{3} \cdot 6 = \frac{24\pi}{3} = 8\pi$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

$P = 3z \qquad Q = 2x \qquad R = y+z$

$\frac{\partial g}{\partial x} = 2 \qquad \frac{\partial g}{\partial y} = 3$

$$\int\int \left(-3z(2) - 2x(3) + y+z\right) dA$$

$$\int_0^1\int_0^1 -6(2x+3y) - 6x + y + 2x+3y \; dx\,dy$$

$$\int_0^1\int_0^1 -12x - 18y - 6x + y + 2x + 3y$$

$$\int_0^1\int_0^1 -16x - 14y \; dx\,dy$$

$$= \int_0^1 \left(-8x^2 - 7y^2 \Big|_0^1\right) dy \qquad -8-7$$

$$= -15y\Big|_0^1 = -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

$4 - 2(2x+y)^{-1}$

---

ans. $(-\frac{1}{4}, 1)$ local min

$2(2x+y)^{-2} \cdot 2$

$f_x = 4 - \frac{1}{2x+y} \cdot 2 = 4 - \frac{2}{2x+y}$

$f_y = 2y - \frac{1}{2x+y}$

$f_{xx} = \frac{4}{(2x+y)^2}$

$f_{yy} = 2 + \frac{1}{(2x+y)^2}$

$f_{xy} = \frac{2}{(2x+y)^2}$

$2y - (2x+y)^{-1}$

$2 + (2x+y)^{-2}$

$4 - 2(2x+1)^{-1}$

$4 - \frac{2}{2x+y} = 0$

$2y - \frac{1}{2x+y} = 0$

$2 - \frac{1}{2x+y} = 0$

$2 = 2y$

$y = 1$

$4 - \frac{2}{2x+1} = 0$

$\frac{4}{1} = \frac{2}{2x+1}$

$8x + 4 = 2$

$8x = -2$

$x = -\frac{1}{4}$

$(-\frac{1}{4}, 1)$

$D = \overset{>0}{16} \cdot 6 - 64 = 32 > 0$

$(-\frac{1}{4}, 1)$ local min

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad . \qquad (2x^2+3y^2+z^2)^{1/2}$$

---

ans. $3.000\overline{3}$

$$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_z = \frac{1}{\cancel{2}}(2x^2+3y^2+z^2)^{-1/2} \cdot \cancel{2}z = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

$$f(1,1,2) = \sqrt{2+3+4} = \sqrt{9} = 3$$

$$f_x(1,1,2) = 2/3$$

$$f_y(1,1,2) = 1$$

$$f_z(1,1,2) = 2/3$$

$$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$$

$$= 3 + \frac{2}{3}(1.001-1) + (0.999-1) + \frac{2}{3}(2.001-2)$$

$$= 3.000\overline{3}$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

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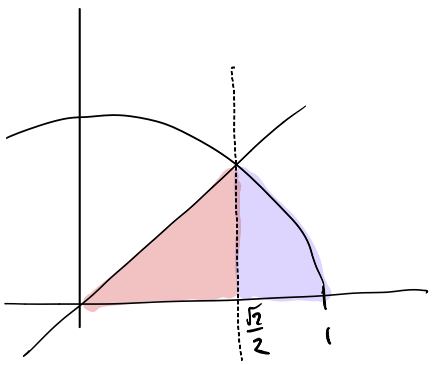
ans.



$0 < r < 1$

$0 < \theta$ $\pi/4$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

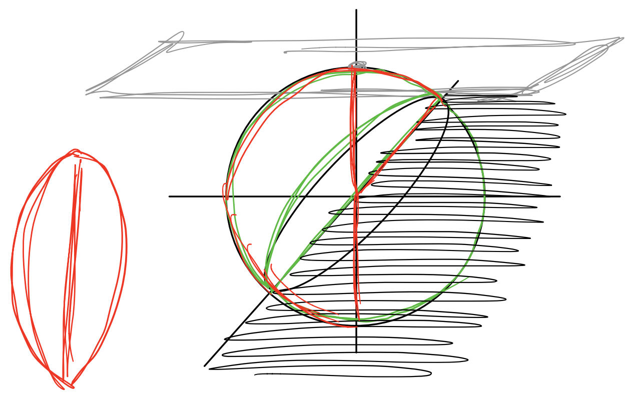
---

ans. $\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{2} \rho^6 \sin^4\phi \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\theta\, d\phi$

$0 < \rho < 2$

$\pi/2 < \theta < \pi$

$0 < \phi < \pi$



$$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{2} (\rho \sin\phi \cos\theta)^2 \cdot \rho \sin\phi \sin\theta \cdot \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$\rho^2 \sin^2\phi \cos^2\theta \cdot \rho$

$$\int_0^{\pi} \int_{\pi/2}^{\pi} \int_0^{2} \rho^6 \sin^4\phi \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\theta\, d\phi$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

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ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$\|r'(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t} = \sqrt{9(\cos^2 t + \sin^2 t)} = \sqrt{9} = 3$

$\langle 0, 3\cos t, -3\sin t \rangle \times \langle 0, -3\sin t, -3\cos t \rangle$

$$= \begin{vmatrix} i & j & k \\ 0 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix} = (-9\cos^2 t + 9\sin^2 t)i - (0)j + (0)K$$

$= \langle -9, 0, 0 \rangle \rightarrow \sqrt{81 + 0 + 0} = \sqrt{81} = 9$

$\kappa(\pi/3) = \frac{9}{3^3} = \frac{9}{27} = \frac{1}{3}$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans.

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16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \qquad .$$

at the point $(1,1,1)$.

---

ans. 14

$$\nabla f = \langle y^2z^3, \ 2xyz^3, \ 3xy^2z^2 \rangle$$

$$\nabla g = \langle 1, \ 2y, \ 3z^2 \rangle$$

$$\langle y^2z^3, \ 2xyz^3, \ 3xy^2z^2 \rangle \cdot \langle 1, \ 2y, \ 3z^2 \rangle$$

$$= y^2z^3 + 4xy^2z^3 + 9xy^2z^4$$

@ $(1,1,1)$: $\quad 1 + 4 + 9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad . \quad \frac{(x+y)(x+y)}{(x+y)-z-w} - \frac{(z+w)(z+w)}{}$$

---

ans.

$$\lim_{xyzw \to t,0,0,0} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{t^2}{t} = t$$

$$\lim_{xyzw \to 0,t,0,0} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{t^2}{t} = t$$

$$\lim_{xyzw \to 0,0,t,0} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{-t^2}{-t} = t$$

$$\lim_{xyzw \to 0,0,0,t} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{-t^2}{-t} = t$$

Limit exists.