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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 18
2. $\int_0^{\frac{1}{2}}\int_{\frac{1}{4}}^1 f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^1\int_{y^2}^1 f(x,y)\,dx\,dy$
3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$
4. $3\hat{i} - 6\hat{j} + 9$
5. A: $\frac{\pi}{3}$ rad B: $\frac{\pi}{3}$ rad C: $\frac{\pi}{3}$ rad
6. $-4\sqrt{3}$
7. $6(\cos^2(1) - \sin^2(1))$
8. 8π
9. -15
10. $(\frac{3}{4}, -1)$ - saddle point
11. 3.006
12. $\frac{\sqrt{2}}{6}$
13. $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^2 r^6 \sin^4\phi \cos\phi \sin\theta \cos^2\theta \,dr\,d\theta\,d\phi$
14. $\frac{1}{3}$
15. $\int_0^1\int_0^v \sqrt{4v^4 + 16u^2v^2 + 4u^4}\,du\,dv$
16. 14
17. 0

Sign the following declaration:

I IM Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Umyz

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx + (sin\,(e^{\cos y}) + 11x)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

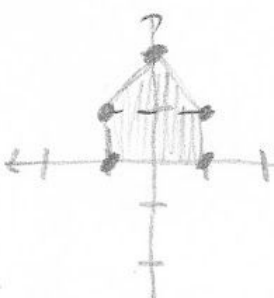
ans. 18

We can use Green's Theorem to simplify the integral:

$\int_C P\,dx + Q\,dy = \iint_D Q_x - P_y \, dA$

$Q_x = 11 \quad P_y = 5 \rightarrow \iint_D 11-5 \, dA = \iint_D 6 \, dA$

The area integral of a constant is the area of the region times that constant. So, let's draw our region:



$\text{Area}_{\triangle} = \text{Area}_{\square} + \text{Area}_{\triangle} = (1 \cdot 2) + \frac{1}{2}(1 \cdot 2) =$
$= 2 + 1 = 3 \rightarrow 3 \cdot 6 = \boxed{18}$

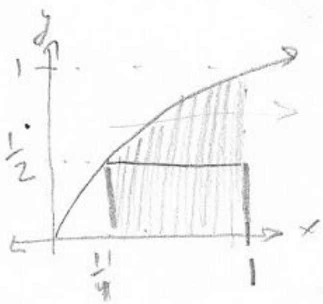
2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, \overset{dy\,dx}{\cancel{dx\,dy}} \quad .$$

---

ans. $\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

Sketch of the region:



We need to split this into 2 regions:
Region 1: $[\frac{1}{4} : 1] \times [0 : \frac{1}{2}]$
Region 2: $y^2 \leq x \leq 1$, $\frac{1}{2} \leq y \leq 1$

Our final integrals are:

$$\boxed{\int_0^{\frac{1}{2}} \int_{\frac{1}{4}}^{1} f(x,y)\,dx\,dy + \int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy}$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

First, check that the point makes sense:

$2\cos(\frac{\pi}{3}) + 4\cos(\frac{\pi}{3}) + 8\cos(\frac{\pi}{3}) = 2(\frac{1}{2}) + 4(\frac{1}{2}) + 8(\frac{1}{2}) = 1+2+4 = 7 = 7$ ✓

We would need to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ for the plane's equation. First, take the derivative of both sides w.r.t. x (treat y as constant, and $z = f(x,y)$):

$-2\sin(x+y) - 4\sin(x+z)(1+\frac{\partial z}{\partial x}) - 8\sin(y+z)(\frac{\partial z}{\partial x}) = 0$

$-4\frac{\partial z}{\partial x}\sin(x+z) - 8\frac{\partial z}{\partial x}\sin(y+z) = 2\sin(x+y) + 4\sin(x+z)$

$\frac{\partial z}{\partial x} = \frac{2\sin(x+y) + 4\sin(x+z)}{-4\sin(x+z) - 8\sin(y+z)} \to (\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) \to \frac{2\sin(\frac{\pi}{3}) + 4\sin(\frac{\pi}{3})}{-4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})} =$

$= \frac{\sqrt{3} + 2\sqrt{3}}{-2\sqrt{3} - 4\sqrt{3}} = \frac{3\sqrt{3}}{-6\sqrt{3}} = -\frac{1}{2}$

Repeat w.r.t. y to find $\frac{\partial z}{\partial y}$:

$-2\sin(x+y) - 4\sin(x+z)(\frac{\partial z}{\partial y}) - 8\sin(y+z)(1+\frac{\partial z}{\partial y}) = 0$

$-4\sin(x+z)(\frac{\partial z}{\partial y}) - 8\sin(y+z)(\frac{\partial z}{\partial y}) = 2\sin(x+y) + 8\sin(y+z)$

$\frac{\partial z}{\partial y} = \frac{2\sin(x+y) + 8\sin(y+z)}{-4\sin(x+z) - 8\sin(y+z)} \to (\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) \to \frac{2\sin(\frac{\pi}{3}) + 8\sin(\frac{\pi}{3})}{-4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})} =$

$= \frac{\sqrt{3} + 4\sqrt{3}}{-2\sqrt{3} - 4\sqrt{3}} = \frac{5\sqrt{3}}{-6\sqrt{3}} = -\frac{5}{6}$

The equation for our tangent plane is:

$z - \frac{\pi}{6} = -\frac{1}{2}(x - \frac{\pi}{6}) - \frac{5}{6}(y - \frac{\pi}{6})$

$z = -\frac{1}{2}x + \frac{\pi}{12} - \frac{5}{6}y + \frac{5\pi}{36} + \frac{\pi}{6} \to \boxed{z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $3\hat{i} - 6\hat{j} + 9\hat{k}$

We can use this property of cross-products:

$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$(a+b+c) \times 2a - (a+b+c) \times (b) + (a+b+c) \times (3c) =$

$= a \times 2a + b \times 2a + c \times 2a - a \times b - b \times b - c \times b + a \times 3c + b \times 3c + c \times 3c =$

$= 0 + 2(b \times a) + 2(c \times a) - (a \times b) - 0 - (c \times b) + 3(a \times c) + 3(b \times c) + 0 =$

$= -2(a \times b) - 2(a \times c) - (\hat{i} + \hat{j} - \hat{k}) + (b \times c) + 3(2\hat{i} + \hat{j} + 2\hat{k}) + 3(\hat{i} - \hat{j} + \hat{k}) =$

$= -2(\hat{i} + \hat{j} - \hat{k}) - 2(2\hat{i} + \hat{j} + 2\hat{k}) - (\hat{i} + \hat{j} - \hat{k}) + (\hat{i} - \hat{j} + \hat{k}) + (6\hat{i} + 3\hat{j} + 6\hat{k}) + (3\hat{i} - 3\hat{j} + 3\hat{k}) =$

$= (-2 \overset{-6}{-4} \overset{-7}{-1} \overset{-6}{+1} + 6 + 3)\hat{i} + (-2 \overset{-4}{-2} \overset{-5}{-1} \overset{-6}{-1} \overset{-3}{+3} - 3)\hat{j} + (2 \overset{-2}{-4} \overset{-1}{+1} \overset{0}{+1} + 6 + 3)\hat{k} =$

$= \boxed{3\hat{i} - 6\hat{j} + 9\hat{k}}$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;

The formula for the angle between two vectors is:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)$$

The vectors that surround angle A are:

$\vec{u} = B - A = \langle 1,0,1\rangle \quad \vec{v} = C - A = \langle 1,1,0\rangle$

$$\theta_A = \cos^{-1}\left(\frac{1+0+0}{\sqrt{1+1}\cdot\sqrt{1+1}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}\text{ rad}}$$

The vectors that surround angle B are:

$\vec{u} = A - B = \langle -1,0,-1\rangle \quad \vec{v} = C - B = \langle 0,1,-1\rangle$

$$\theta_B = \cos^{-1}\left(\frac{0+0+1}{\sqrt{1+1}\cdot\sqrt{1+1}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}\text{ rad}}$$

The vectors that surround angle C are:

$\vec{u} = A - C = \langle -1,-1,0\rangle \quad \vec{v} = B - C = \langle 0,-1,1\rangle$

$$\theta_C = \cos^{-1}\left(\frac{0+1+0}{\sqrt{1+1}\cdot\sqrt{1+1}}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \boxed{\frac{\pi}{3}\text{ rad}}$$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

---

ans. $-4\sqrt{3}$

The directional derivative is the dot product of the function's gradient vector and the unit vector of the path.

First, find the gradient:

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle 3x^2+yz,\ 3y^2+xz,\ 3z^2+xy \rangle$

Find its value at the point $(1,1,1)$:

$\nabla f(1,1,1) = \langle 3+1,\ 3+1,\ 3+1 \rangle = \langle 4,4,4 \rangle$

Next, find the unit vector of the path:

$\vec{v} = \langle -1-1, -1-1, -1-1 \rangle = \langle -2,-2,-2 \rangle$

$\vec{u} = \dfrac{\langle -2,-2,-2 \rangle}{\sqrt{4+4+4}} = \dfrac{\langle -2,-2,-2 \rangle}{\sqrt{12}} = \dfrac{\langle -2,-2,-2 \rangle}{2\sqrt{3}} = \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$

Take the dot product:

$\nabla f \cdot \vec{u} = -\dfrac{4}{\sqrt{3}} - \dfrac{4}{\sqrt{3}} - \dfrac{4}{\sqrt{3}} = \dfrac{-12}{\sqrt{3}} = \dfrac{-12\sqrt{3}}{3} = -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6(\cos(1)^2 - \sin(1)^2)$

The formula for the Chain Rule is:

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$$

So, find all of the partial derivatives needed:

$$\frac{\partial g}{\partial x} = 6x \quad \frac{\partial g}{\partial y} = -6y \quad \frac{\partial x}{\partial u} = e^u \cos v \quad \frac{\partial y}{\partial u} = e^u \sin v$$

$$\frac{\partial g}{\partial u} = (6x)(e^u \cos v) + (-6y)(e^u \sin v) = (6e^u \cos v)(e^u \cos v) + (-6e^u \sin v)(e^u \sin v) =$$

$$= 6e^{2u} \cos(v)^2 - 6e^{2u} \sin(v)^2$$

Plug in $(u, v) = (0, 1)$:

$$\frac{\partial g}{\partial u}(0,1) = 6e^0 \cos(1)^2 - 6e^0 \sin(1)^2 = 6(\cos(1)^2 - \sin(1)^2)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

We can use the Divergence theorem to calculate the integral:

$\iint_S \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div}(\vec{F}) \cdot dV$

First, find the divergence of the vector field:

$\operatorname{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 3 - 2 + 5 = 6$

So, our new integral is:

$\iiint_E 6\, dV$

The volume integral of a constant is the volume of the region times the constant. Our region is $\frac{1}{8}$ of a sphere of radius 2. So, our surface integral is equal to:

$6 \cdot \left(\frac{4}{3}\right)\pi (2)^3 \cdot \left(\frac{1}{8}\right) = \frac{\cancel{6}^{2} \cdot 4 \cdot 8}{\cancel{3} \cdot 8}\pi = \boxed{8\pi}$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,, 2x\,, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

The surface integral of a vector field with the surface z with upward pointing normal can be calculated like this:

$$\iint_D -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R\, dA$$

So first, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$:

$$\frac{\partial z}{\partial x} = 2 \qquad \frac{\partial z}{\partial y} = 3$$

Plug in the needed values, and the x and y bounds given in the question:

$$\int_0^1\int_0^1 -3z(2) - 2x(3) + y + z \, dy\, dx = \int_0^1\int_0^1 -6(2x+3y) - 6x + y + 2x + 3y \, dy\, dx =$$

$$= \int_0^1\int_0^1 -12x - 18y - 6x + y + 2x + 3y \, dy\, dx = \int_0^1\int_0^1 -16x - 14y\, dy\, dx =$$

$$= \int_0^1 -16xy - 7y^2\Big|_0^1 dx = \int_0^1 -16x - 7\, dx = -8x^2 - 7x\Big|_0^1 = -8 - 7 = \boxed{-15}$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. $(\frac{3}{4}, -1) \rightarrow$ saddle point

The critical points occur when $f_x = f_y = 0$:

$f_x = 4 + (2)\left(\frac{-1}{2x+y}\right) = 4 - \frac{2}{2x+y} = 0$

$f_y = -2y - \frac{1}{2x+y} = 0$

$f_x - 2f_y = 4 + 4y - \frac{2}{2x+y} + \frac{2}{2x+y} = 4 + 4y = 0 \rightarrow 4y = -4 \rightarrow \underline{y = -1}$

$4 - \frac{2}{2x-1} = 0 \rightarrow \frac{2}{2x-1} = 4 \rightarrow 4(2x-1) = 2 \rightarrow 2x - 1 = \frac{1}{2} \rightarrow 2x = \frac{3}{2} \rightarrow \underline{x = \frac{3}{4}}$

Our critical point is $(\frac{3}{4}, -1)$

We can use the 2nd derivative test to see if it is a local max, local min, or saddle point:

$D = f_{xx}(x_0,y_0) \cdot f_{yy}(x_0,y_0) - f_{xy}(x_0,y_0)^2$

If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) > 0$, (x_0,y_0) is a local minimum

If $D(x_0,y_0) > 0$ and $f_{xx}(x_0,y_0) < 0$, (x_0,y_0) is a local maximum

If $D(x_0,y_0) < 0$, (x_0,y_0) is a saddle point

If $D(x_0,y_0) = 0$, it is inconclusive

$f_{xx} = 0 - \frac{(2x+y)(0) - (2)(2)}{(2x+y)^2} = \frac{4}{(2x+y)^2}$ $\quad f_{yy} = -2 - \frac{(2x+y)(0) - (1)(1)}{(2x+y)^2} = -2 + \frac{1}{(2x+y)^2}$

$f_{xy} = 0 - \frac{(2x+y)(0) - (2)(1)}{(2x+y)^2} = \frac{2}{(2x+y)^2} \Rightarrow D(\frac{3}{4}, -1) = \frac{4}{(\frac{3}{2}-1)^2}\left(-2 + \frac{1}{(\frac{3}{2}-1)^2}\right) - \left(\frac{2}{(\frac{3}{2}-1)^2}\right)^2 =$

$= \frac{4}{\frac{1}{4}}(-2+4) - \left(\frac{2}{\frac{1}{4}}\right)^2 = 16(2) - 64 = -32$

$D(\frac{3}{4}, -1) < 0, \rightarrow \boxed{(\frac{3}{4}, -1) \text{ is a saddle point}}$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.006

The formula for linearization is:

$L(x, y, z) = f_x(x - x_0) + f_y(y - y_0) + f_z(z - z_0) + f(x, y, z)$

First, find all of the needed partial derivatives:

$f_x = (4x)\sqrt{2x^2 + 3y^2 + z^2} \rightarrow f_x(1,1,2) = 4\sqrt{2+3+2^2} = 4\sqrt{9} = 12$

$f_y = (6y)\sqrt{2x^2 + 3y^2 + z^2} \rightarrow f_y(1,1,2) = 6\sqrt{2+3+2^2} = 6\sqrt{9} = 18$

$f_z = (2z)\sqrt{2x^2 + 3y^2 + z^2} \rightarrow f_z(1,1,2) = 4\sqrt{2+3+2^2} = 4\sqrt{9} = 12$

Plug in $(1,1,2)$ into $f(x, y, z)$:

$f(1,1,2) = \sqrt{2+3+2^2} = \sqrt{2+3+4} = \sqrt{9} = 3$

So, our linearization formula would be:

$L(x, y, z) = 12(x-1) + 18(y-1) + 12(z-2) + 3$

Plug in $(1.001, 0.999, 2.001)$:

$L(1.001, 0.999, 2.001) = 12(0.001) + 18(-0.001) + 12(0.001) + 3 =$

$= \underbrace{0.012 - 0.018}_{-0.006} + 0.012 + 3 = \boxed{3.006}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

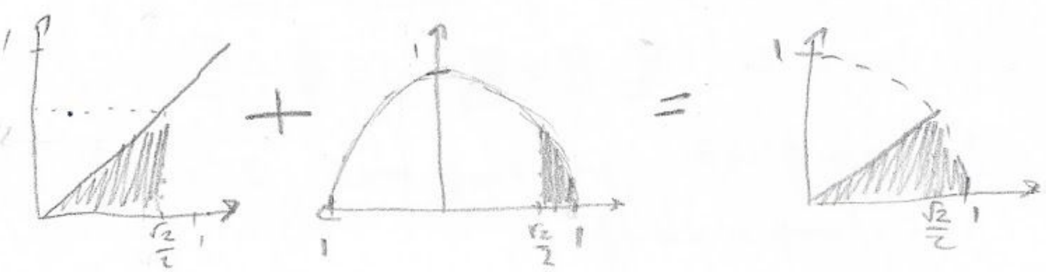
$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $\frac{\sqrt{2}}{6}$

Regions:



Our new region has bounds $0 \le r \le 1$ and $0 \le \theta \le \frac{\pi}{4}$

Our new, combined integral, is:

$$\int_0^{\frac{\pi}{4}} \int_0^1 r^2 \cos\theta \, dr\, d\theta = \int_0^{\frac{\pi}{4}} \frac{r^3}{3} \cos\theta \Big|_0^1 d\theta = \int_0^{\frac{\pi}{4}} \frac{\cos\theta}{3} d\theta =$$

$$= \frac{\sin\theta}{3} \Big|_0^{\frac{\pi}{4}} = \frac{\sin\frac{\pi}{4}}{3} - \frac{\sin(0)}{3} = \frac{\frac{\sqrt{2}}{2}}{3} - 0 = \boxed{\frac{\sqrt{2}}{6}}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2}\int_0^{\sqrt{4-z^2}}\int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates. Do not evaluate.**

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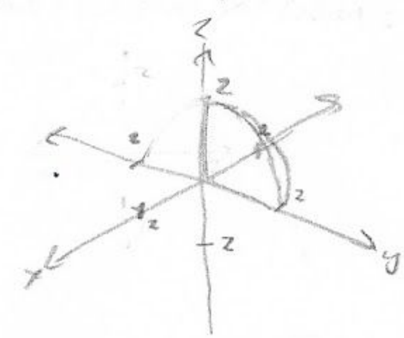
ans. $\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} r^6 \sin^4\phi \cos^2\phi \sin\theta \cos\theta \, dr\, d\theta\, d\phi$

Reason sketch:



Our new r, θ and ϕ bounds are:

$0 \le r \le 2$, $\frac{\pi}{2} \le \theta \le \pi$, $0 \le \phi \le \frac{\pi}{2}$

$x = r\sin\phi\cos\theta$ $\quad y = r\sin\phi\sin\theta$ $\quad z = r\cos\phi$

$dV = r^2 \sin\phi \, dr\, d\theta\, d\phi$

So, our new integral is:

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} r^6 \sin^2\phi \cos^2\theta \sin\phi \sin\theta \cos\phi \sin\phi \, dr\, d\theta\, d\phi =$$

$$= \boxed{\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} r^6 \sin^4\phi \cos\phi \sin\theta \cos^2\theta \, dr\, d\theta\, d\phi}$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $\frac{1}{3}$

The formula for curvature is:

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle \Rightarrow r'\left(\tfrac{\pi}{3}\right) = \langle 0, 3\cos\tfrac{\pi}{3}, -3\sin\tfrac{\pi}{3}\rangle =$$
$$= \langle 0, \tfrac{3}{2}, -\tfrac{3\sqrt{3}}{2}\rangle$$

$$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle \rightarrow r''\left(\tfrac{\pi}{3}\right) = \langle 0, -3\sin\tfrac{\pi}{3}, -3\cos\tfrac{\pi}{3}\rangle =$$
$$= \langle 0, -\tfrac{3\sqrt{3}}{2}, -\tfrac{3}{2}\rangle$$

$$r'\left(\tfrac{\pi}{3}\right) \times r''\left(\tfrac{\pi}{3}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{3}{2} & -\frac{3\sqrt{3}}{2} \\ 0 & -\frac{3\sqrt{3}}{2} & -\frac{3}{2} \end{vmatrix} = \hat{i}\left(-\tfrac{9}{4} - \tfrac{9(3)}{4}\right) - \hat{j}(0) + \hat{k}(0) =$$
$$= \langle -\tfrac{36}{4}, 0, 0\rangle = \langle -9, 0, 0\rangle$$

$$|\langle -9, 0, 0\rangle| = 9$$

$$\left|r'\left(\tfrac{\pi}{3}\right)\right|^3 = \left(\sqrt{0 + \tfrac{9}{4} + \tfrac{9(3)}{4}}\right)^3 = \left(\sqrt{\tfrac{36}{4}}\right)^3 = \left(\sqrt{9}\right)^3 = 3^3 = 27$$

$$\kappa\left(\tfrac{\pi}{3}\right) = \frac{9}{27} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_0^v \sqrt{4v^4 + 16u^2v^2 + 4u^4}\, du\, dv$

$x = u^2 \qquad y = uv \qquad z = v^2$

$\int_C 1\, dS = \int_{v_1}^{v_2} \int_{u_1}^{u_2} \|\vec{r}_u \times \vec{r}_v\|\, dA$

$\vec{r}_u = \langle 2u, v, 0\rangle \qquad \vec{r}_v = \langle 0, u, 2v\rangle$

$\vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = \hat{i}(2v^2) - \hat{j}(4uv) + \hat{k}(2u^2) =$

$= \langle 2v^2, -4uv, 2u^2\rangle$

$\|\vec{r}_u \times \vec{r}_v\| = \sqrt{4v^4 + 16u^2v^2 + 4u^4}$

Our final integral is:

$$\int_0^1 \int_0^v \sqrt{4v^4 + 16u^2v^2 + 4u^4}\, du\, dv$$

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla f(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 1, 2y, 3z^2 \rangle$

$\nabla g(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla f \cdot \nabla g = 1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 = 1 + 4 + 9 = \boxed{14}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

---

ans. 0

Let's try to simplify the limit:

$$\frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{(x+y+z+w)\cancel{(x+y-z-w)}}{\cancel{x+y-z-w}} = x+y+z+w$$

Now, let's plug in $(0,0,0,0)$:

$$x+y+z+w = 0+0+0+0 = \boxed{0}$$