Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: [illegible]

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad .$$

number - or +

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0)$$ closed surface

clockwise

Explain!

ans. -18

1. check if its conservative

$\frac{d}{dy}(\cos(e^{\sin x}) + 5y) = \frac{d}{dx}(\sin(e^{\cos y}) + 11x)$

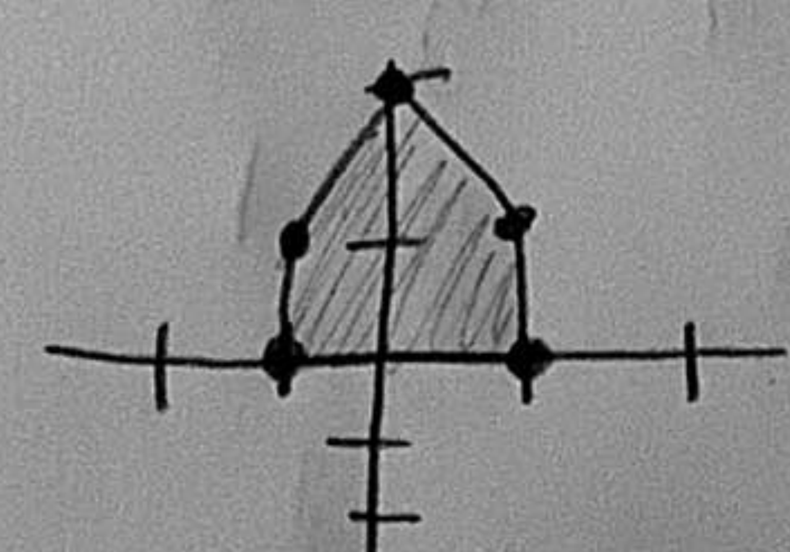
$5 = 11$ fundamental Theorem of Line Integrals does not apply

2. Green Theorem (closed + counter-clockwise)

Multiple by -1 for clockwise

$-1 \iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\, dA = -1\iint_D 11 - 5\, dA$

$-1 \iint 6\, dA$



$-6 \times$ area of region

Area $= 2 + \frac{1}{2}(1)(1) + \frac{1}{2}(1)(1)$

-6×3

Area $= 2 + 1 = 3$

-18

Since it is a closed surface, Green's Theorem can be applied finding the integrand to be 6. Since it is a simple number, it can be multiplied by the regions area to find the integral.

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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_0^1 \int_{y^2}^{2} f(x,y)\,dx\,dy$
3. $z = -2x - \frac{5}{6}y + \frac{14\pi}{36}$
4. $-i - 7j + 7k$
5. $\frac{\pi}{3}$ for all
6. $-4\sqrt{3}$
7. 2.49688
8. 8π
9. $\frac{5}{2}$
10. $(\frac{3}{4}, 1)$ and $(\frac{1}{4}, 0)$
11. 3.00033
12. $\frac{\sqrt{2}}{6}$
13. $\int_0^{\pi/2}\int_{\pi/2}^{\pi/4}\int_0^2 p^6 \sin^4\phi \cdot \cos^2\theta \cdot \sin\theta \cos\phi \, dp\, d\phi\, d\theta$
14. $\rightarrow \frac{1}{3}$
15. $\int\int^{1}\,^{1-v} r(u,v)\sqrt{4u^4+16u^2v^2+4u^2}\,du\,dv$
16. $0 \quad 0 \rightarrow 14$
17. 0

2. (12 points) Change the order of integration

$$\cancel{\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dx\,dy}$$

Emailed correction

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx$$

---

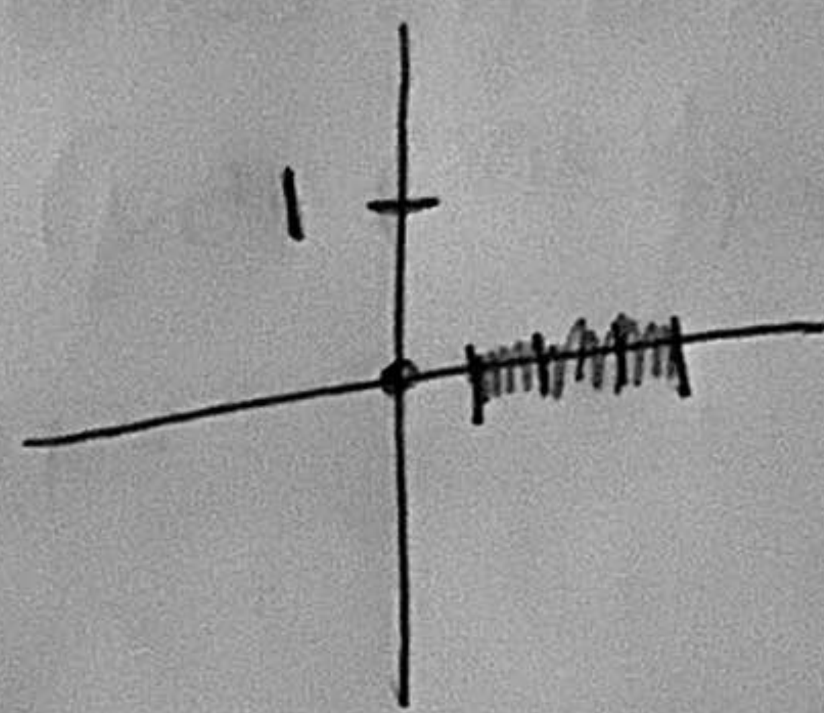
ans. $\displaystyle\int_{0}^{\frac{1}{2}} \int_{y^2}^{1} f(x,y)\,dx\,dy$

$0 \le y \le \sqrt{x}$ $\quad$ $\frac{1}{4} \le x \le 1$

$y^2 \le x$ $\quad$ $y^2 \le x \le 1$

$(\frac{1}{4})^2 = x$ $\quad$ $(1)^2 = x$

$x = \frac{1}{2}$ $\quad$ $x = 1$



$y = \sqrt{\frac{1}{4}}$ $\quad$ $y = \sqrt{1}$

$y = \frac{1}{2}$ $\quad$ $y = 1$

$0 \le y \le \frac{1}{2}$

$\frac{14\sqrt{3}\pi}{6} \cdot \frac{1}{6\sqrt{3}}$

$\frac{14\pi}{36}$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad . \quad -2x - \frac{5}{6}y + \frac{14\pi}{36} = z$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

ans. $z = -2x - \frac{5}{6}y + \frac{14\pi}{36}$

1. Check @ point

$2\cos(\frac{\pi}{6} + \frac{\pi}{6}) + 4\cos(\frac{\pi}{6} + \frac{\pi}{6}) + 8\cos(\frac{\pi}{6} + \frac{\pi}{6}) = 7$

$1 + 2 + 4 = 7$

$7 = 7$ ✓ point exists on plane

$f_x = 2 \cdot -\sin(x+y) \cdot 1 + 4 \cdot -\sin(x+z) \cdot 1 + 0$

$f_x = -2\sin(x+y) - 4\sin(x+z)$

$f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -2\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3}) = \frac{-2\sqrt{3}}{2} - \frac{4\sqrt{3}}{2} = -3\sqrt{3}$

$f_y = -2\sin(x+y) \cdot 1 + 0 + 8\sin(y+z) \cdot 1$

$f_y = -2\sin(x+y) - 8\sin(y+z)$

$f_y(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -2\sin\frac{\pi}{3} - 8\sin\frac{\pi}{3} = \frac{-2\sqrt{3}}{2} - \frac{8\sqrt{3}}{2} = -5\sqrt{3}$

$f_x = -4\sin(x+z) - 8\sin(y+z)$

$f_x(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}) = -4\sin\frac{\pi}{3} - 8\sin\frac{\pi}{3} = \frac{-4\sqrt{3}}{2} - \frac{8\sqrt{3}}{2} = -6\sqrt{3}$

$-3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6}) = 0$

$-3\sqrt{3}x - 5\sqrt{3}y + \frac{3\sqrt{3}\pi}{6} + \frac{5\sqrt{3}\pi}{6} + \frac{6\sqrt{3}\pi}{6} = 6\sqrt{3}z$ divided by $6\sqrt{3}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \quad ?$$

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ans. $-i - 7j + 7k$

$$(a+b+c) \times (2a-b+3c) =$$

$$(a \times 2a - b + 3c) + (b \times 2a - b + 3c) + (c \times 2a - b + 3c)$$

$$\cancelto{0}{a \times 2a} + a \times -b + a \times 3c + b \times 2a + \cancelto{0}{b \times b} + b \times 3c +$$

$$c \times 2a + c \times -b + \cancelto{0}{c \times 3c}$$

$$-1(a \times b) + 3(a \times c) + 2(b \times a) + 3(b \times c) +$$

$$3(c \times a) + -1(c \times b)$$

$$-1(i+j-k) + 3(2i+j+2k) + 2(-i-j+k) +$$

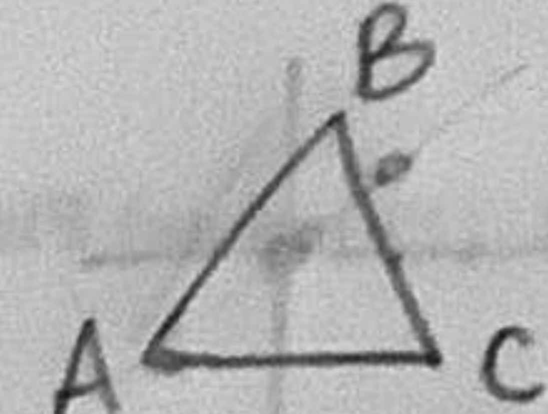
$$3(i-j+k) + 3(-2i-j-2k) + -1(-i+j-k)$$

$$\cancel{-i} \cancel{-j} \cancel{+k} + \cancel{6i} + \cancel{3j} + \cancel{6k} - \cancel{2i} - \cancel{2j} + \cancel{2k} +$$

$$\cancel{3i} - \cancel{3j} + \cancel{3k} - \cancel{6i} - \cancel{3j} - \cancel{6k} \cancel{-} i - \cancel{j} + \cancel{k}$$

$$-i - 7j + 7k$$

5. (12 points) Find the three angles of the triangle ABC where



$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

A·B is angle at A

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ans. The angle at A is: radians ; $\frac{\pi}{3}$

The angle at B is: radians ; $\frac{\pi}{3}$

The angle at C is: radians ; $\frac{\pi}{3}$

$$\cos\theta = \frac{A \cdot B}{(|A||B|)}$$

$$\overline{AB} = (1,0,1) - (0,0,0) = \langle 1,0,1 \rangle \quad |\overline{AB}| = \sqrt{2}$$

$$\overline{AC} = (1,1,0) - (0,0,0) = \langle 1,1,0 \rangle \quad |\overline{AC}| = \sqrt{2}$$

$$\frac{AB \cdot AC}{2} = \frac{\langle 1,0,1 \rangle \cdot \langle 1,1,0 \rangle}{2} = \frac{1+0+0}{2} = \frac{1}{2}$$

$$\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}$$

$$\overline{BA} = (0,0,0) - (1,0,1) = \langle -1,0,-1 \rangle \quad |\overline{BA}| = \sqrt{2}$$

$$\overline{BC} = (1,1,0) - (1,0,1) = \langle 0,1,-1 \rangle \quad |\overline{BC}| = \sqrt{2}$$

$$\frac{\overline{BA} \cdot \overline{BC}}{2} = \frac{\langle -1,0,-1) \cdot \langle 0,1,-1 \rangle}{2} = \frac{0+0+1}{2} = \frac{1}{2}$$

$$\cos^{-1}\left(\tfrac{1}{2}\right) = \frac{\pi}{3}$$

$$2\pi - \frac{\pi}{3} - \frac{\pi}{3} = \frac{\pi}{3}$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

ans. $-4\sqrt{3}$

1. gradient

$\nabla f = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle$

2. Find unit vector

$|\langle -1, -1, -1 \rangle| = \sqrt{-1^2 + -1^2 + -1^2} = \sqrt{3}$

$u = \langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \rangle$

$\nabla f(1,1,1) = \langle 3+1, 3+1, 3+1 \rangle$

$= \langle 4, 4, 4 \rangle$

$\nabla f \cdot u = \langle 4, 4, 4 \rangle \cdot \langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \rangle$

$= \frac{-4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} = \frac{-12}{\sqrt{3}} = \frac{-12\sqrt{3}}{3}$

$= -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

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ans. 2.49688

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial g}{\partial x} = 6x \qquad \frac{\partial g}{\partial y} = -6y \qquad \frac{\partial x}{\partial u} = e^u \cos v \qquad \frac{\partial y}{\partial u} = e^u \sin v$$

at $(u, v) = (0, 1)$

$$\frac{\partial g}{\partial x} = 6(e^u \cos v) \quad \frac{\partial g}{\partial x} = -6(e^u \sin v) \quad \frac{\partial x}{\partial u} = e^0 \cos 1 \quad \frac{\partial y}{\partial u} = e^0 \sin 1$$

$$\frac{\partial g}{\partial x} = 6(e^0 \cos 1) \quad \frac{\partial g}{\partial x} = -6(e^0 \sin 1)$$

$$= 6\cos 1 \cdot \cos 1 - 6 \sin 1 \cdot \sin 1$$

$$= (6\cos^2 1) - (6 \sin^2 1)$$

$$= 2.49688$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3x + \cos(y^3 + yz),\ -2y + e^{x+z^2},\ 5z + \sin(xy^3 + e^x) \rangle \quad ,$$

and S is the closed surface in 3D space bounding the region 

$$\{(x,y,z) : x^2 + y^2 + z^2 < 4 \ \ and \ \ x > 0 \ \ and \ \ y < 0 \ \ and \ \ z > 0\} \ .$$

$0 \le \rho \le 2 \qquad \frac{3\pi}{2} \le \theta \le 2\pi \qquad 0 \le \phi \le \frac{\pi}{2}$

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ans. 8π

$\text{div}(F) = \langle 3 - 2 + 5 \rangle$

$$\iiint_D 6 \, dV$$

$$\int_{3\pi/2}^{2\pi} \int_0^{\pi/2} \int_0^2 6\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_{3\pi/2}^{2\pi} \int_0^{\pi/2} \frac{\rho^3}{3} \Big|_0^2 6 \sin\phi \, d\phi \, d\theta$$

$$16 \int_{3\pi/2}^{2\pi} -\cos\phi \Big|_0^{\pi/2} d\theta$$

$$\int_{3\pi/2}^{2\pi} 16 \, d\theta = 16\left(2\pi - \frac{3\pi}{2}\right) = 8\pi$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\,2x\,,\,y+z\rangle\ ,$$

and S is the oriented surface

$$z = 2x+3y\quad ,\quad 0<x<1,\quad 0<y<1\ ,$$

with **upward pointing** normal.

ans. $\frac{5}{2}$

-1

$\text{div}(F) = 0+0+1$

$\iiint_D 1\, dV$

$\int_0^1\int_0^1\int_0^{2x+3y} 1\, dz\, dy\, dx$

$\int_0^1\int_0^1 2x+3y\, dy\, dx$

$\int_0^1 2xy + \frac{3y^2}{2}\Big|_0^1 dy$

$\frac{2x^2}{2} + \frac{3}{2}\Big|_0^1 = \frac{1^2}{2}+\frac{3}{2} = \frac{5}{2}$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

ans. $(\frac{3}{4}, 1)$ and $(\frac{1}{4}, 0)$

$f_x = 4 - \frac{2}{2x+y}$

$0 = 4 - \frac{2}{2x+y}$

$-2y - \frac{-1}{2x+y} = 4 - \frac{2}{2x+y}$

$-2y = 4 - \frac{1}{2x+y}$

$y = -2 + \frac{1}{2x+y}$

$(\frac{1}{4}, 0)$

$f_{xx} = 2\left(\frac{1}{(2x+y)^2} \cdot 2\right)$

$f_{yy} = -2 + \frac{1}{(2x+y)^2}$

$f_{xy} = 2\left(\frac{1}{(2x+y)^2}\right)$

$f_y = -2y - \frac{1}{2x+y}$

$0 = -2y - \frac{1}{2x+y}$

$\frac{1}{2x+y} = -2y$

$1 = -2y(2x+y)$

$1 = -4xy - 2y^2$

$1 + 2y^2 = -4xy$

$\frac{-2y^2+1}{4y} = x$

$-4 = \frac{-2}{2\left(\frac{-2y^2-1}{4y}\right) + y}$

$-4 = \frac{-2}{\frac{-4y^2-2}{4y} + \frac{4y^2}{4y}}$

$-4 = \frac{-8y}{-2}$

$8 = -8y$

$-1 = y$

$x = \frac{-2(-1^2) - 1}{4(-1)}$

$x = \frac{-3}{-4} = \frac{3}{4}$ $\quad (\frac{3}{4}, -1)$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad .$$

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ans. 3.0003

$f(1,1,2) = \sqrt{2+3+4} = \sqrt{9} = 3$

at $(1,1,2)$

$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x = \frac{1}{2}(9)^{-1/2} \cdot 4(1) = \frac{4}{6}$

$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y = \frac{1}{2}(9)^{-1/2} \cdot 6(1) = 1$

$f_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 2z = \frac{1}{2}(9)^{-1/2} \cdot 4 = \frac{4}{6}$

$L(x,y) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$

$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(.001) + (-.001) + \frac{2}{3}(.001)$

$= 3.000\overline{3}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain!

ans. $\frac{\sqrt{2}}{6}$

$0 \le x \le \frac{\sqrt{2}}{2} \quad 0 \le y \le x \qquad \frac{\sqrt{2}}{2} \le x \le 1 \quad 0 \le y \le \sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$

$r = \sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2}$

$0 \le r \le 1$

$\sqrt{1-\frac{2}{4}} = \sqrt{\frac{1}{2}}$

$r = \sqrt{1} = 1$

$\int_0^{\pi/4}\int_0^1 r\cos\theta \, r \, dr \, d\theta$

Since they are the same integrand and have the same r they can be combined to one integral.

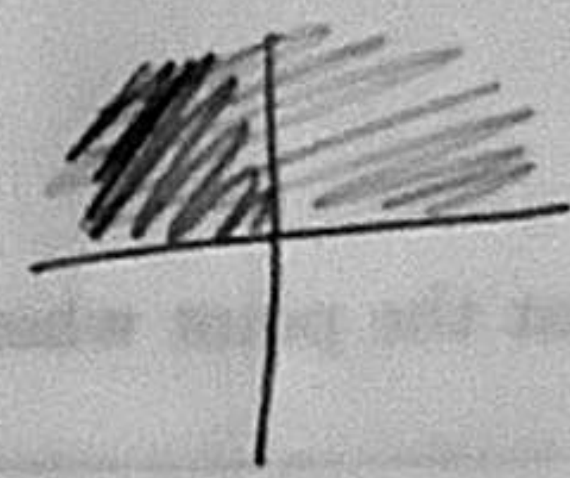
$\int_0^{\pi/4} \frac{r^3}{3}\Big|_0^1 \cos\theta \, d\theta$

$\frac{1}{3} \sin\theta \Big|_0^{\pi/4}$

$\frac{1}{3} \cdot \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{6}$

13. (12 points) Convert the triple iterated integral

$$\int_0^{2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$



to spherical coordinates. Do not evaluate.

ans. $\int_0^{\pi/2}\int_{\pi/2}^{\pi}\int_0^{2} \rho^6 \sin^4\phi \cdot \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\phi\, d\theta$

$0 \le z \le 2$ $\quad$ $0 \le y \le \sqrt{4-z^2}$ $\quad$ $0 \le x \le -\sqrt{4-z^2-y^2}$

z is not neg $\quad$ $0 \le y \le \sqrt{4-0}$ $\quad$ $\rho^2 = x^2+y^2+z^2$

$0 \le \phi \le \frac{\pi}{2}$ $\quad$ $0 \le y \le 2$ $\quad$ $x^2 = 4-z^2-y^2$

$x = -\sqrt{4-4}$ $\quad$ $\rho = 2$

$x = 0$ $\quad$ $0 \le \rho \le 2$

$x = -\sqrt{4-0-0}$

$x = -2$

$\frac{\pi}{2} \le \theta \le \pi$

$$\int_0^{\pi/2}\int_{\frac{\pi}{2}}^{\pi}\int_0^{2} (\rho^2 \sin^2\phi \cos^2\theta)(\rho \sin\phi \sin\theta)(\rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_{\pi/2}^{\pi}\int_0^{2} \rho^6 \sin^4\phi \cdot \cos^2\theta \sin\theta \cos\phi \, d\rho\, d\phi\, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle = \langle 0, 3\cos\frac{\pi}{3}, -3\sin\frac{\pi}{3}\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle = \langle 0, -3\sin\frac{\pi}{3}, -3\cos\frac{\pi}{3}\rangle$

$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & \frac{3}{2} & -\frac{3\sqrt{3}}{2} \\ 0 & -\frac{3\sqrt{3}}{2} & \frac{-3}{2} \end{vmatrix} =$

$= \left(\frac{-9}{4} - \frac{27}{4}\right)i - 0j + 0k \quad \langle -9, 0, 0\rangle$

$|r'(t) \times r''(t)| = \sqrt{-9^2 + 0^2 + 0^2} = 9$

$|r'(t)| = \sqrt{0^2 + \left(\frac{3}{2}\right)^2 + \left(-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{\frac{9}{4} + \frac{27}{4}} = \sqrt{9} = 3$

$\kappa\left(\frac{\pi}{3}\right) = \frac{9}{3^3} = \frac{9}{27} = \frac{1}{3}$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \, , \, uv \, , \, v^2 \rangle \quad , \quad 0 < u < v < 1 \; .$$

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ans. $\int_0^1 \int_0^{1-v} r(u,v) \sqrt{4v^4 + 16u^2v^2 + 4u^2} \, du \, dv$

$r_u = \langle 2u, v, 0 \rangle$

$r_v = \langle 0, u, 2v \rangle$

$$|r_u \times r_v| = \begin{vmatrix} i & j & k \\ 2u & v & 0 \\ 0 & u & 2v \end{vmatrix} = (2v^2 - 0)i - (4uv)j + 2u^2k$$

$\langle 2v^2, -4uv, 2u^2 \rangle \qquad \sqrt{4v^4 + 16u^2v^2 + 4u^2}$

$$\int_0^1 \int_0^{1-v} r(u,v) \cdot \sqrt{4v^2 + 16u^2v^2 + 4u^2}$$

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14

$grad(f) = \langle y^2z^3, 2yxz^3, 3z^2xy^2 \rangle$

$grad(g) = \langle x, 2y, 3z^2 \rangle$

$grad\,f \cdot grad\,g = \langle y^2z^3, 2yxz^3, 3z^2xy^2 \rangle \cdot \langle x, 2y, 3z^2 \rangle$

$= xy^2z^3 + 4y^2xz^3 + 9z^4xy^2$

at $(1,1,1)$

$= (1)(1)(1) + 4(1)(1)(1) + 9(1)(1)(1)$

$= 1 + 4 + 9$

$= 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. 0

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{0-0}{0} = \frac{0}{0}$$

$$a^2-b^2 = (x+y)^2-(z+w)^2$$

$$(a-b)(a+b) = [(x+y)-(z+w)]\cdot[(x+y)+(z+w)]$$

$$=(x+y-z-w)(x+y+z+w)$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{\cancel{(x+y-z-w)}(x+y+z+w)}{\cancel{x+y-z-w}}$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} x+y+z+w = 0+0+0+0=0$$