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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$
3. $-\frac{1}{2}x - \frac{5}{6}y + \frac{2\pi}{3}$
4. $3i - 6j + 9K$
5. $\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}$
6. $-\frac{12}{\sqrt{3}}$
7. $6\cos(2)$
8. 64π
9. -15
10. $\left(\frac{3}{4}, -1\right)$ saddle point
11. 3.000333
12. $\sqrt{2}/6$
13. $\int_{-\frac{\pi}{2}}^{0} \int_0^2 \int_0^2 (r^4 \cos^2\theta \sin\theta\, z)\, dz\,dr\,d\theta$
14. $1/3$
15. $\int_0^1 \int_u^1 (2u^2+2v^2)\,dv\,du$
16. 14
17. 0

Sign the following declaration:

I Daniel Gameiro Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Daniel T. Gameiro 12/15/2020

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\, dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\, dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\, dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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ans. -18

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using Greene's theorem:

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

$$\iint_D (11-5)\, dA = 6 \cdot \text{area}(D)$$

Area of $D = 3$

result: $6 \cdot 3 \cdot (-1)$ since its clockwise

$= -18$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, dy\, dx \qquad .$$

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ans. $\int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$

$y^2 \leq x \leq 1 \qquad\qquad 0 \leq y \leq 1$

$$\int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{3}$

$$-2\sin(x+y) - 4\sin(x+z) - 4\sin(x+z)\frac{\partial z}{\partial x} - 8\sin(y+z)\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = -\frac{2\sin(x+y) + 4\sin(x+z)}{4\sin(x+z) + 8\sin(y+z)}$$

at $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) = -\frac{1}{2}$

$$-2\sin(x+y) - 4\sin(x+z)\frac{\partial z}{\partial y} - 8\sin(y+z) - 8\sin(y+z)\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = -\frac{2\sin(x+y) + 8\sin(y+z)}{4\sin(x+z) + 8\sin(y+z)}$$

at $\left(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}\right) = -\frac{5}{6}$

tangent plane: $z - \frac{\pi}{6} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right) - \frac{5}{6}\left(y - \frac{\pi}{6}\right)$

$$z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{3}$$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $3i - 6j + 9k$

$$(a+b+c) \times (2a - b + 3c)$$

$$= 2(a \times a) - (a \times b) + 3(a \times c) + 2(b \times a) - (b \times b)$$
$$+ 3(b \times c) + 2(c \times a) - (c \times b) + 3(c \times c)$$

$$= -3(a \times b) + (a \times c) + 4(b \times c)$$

$$= -3(i + j - k) + (2i + j + 2k) + 4(i - j + k)$$

$$= 3i - 6j + 9k$$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;

$AB = \langle 1, 0, 1 \rangle \quad AC = \langle 1, 1, 0 \rangle \quad BC = \langle 0, 1, -1 \rangle$

$$\theta_A = \cos^{-1}\left(\frac{AB \cdot AC}{|AB|\,|AC|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\theta_B = \cos^{-1}\left(\frac{BC \cdot BA}{|BC|\,|BA|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

$$\theta_C = \cos^{-1}\left(\frac{CA \cdot CB}{|CA|\,|CA|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

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ans. $-12/\sqrt{3}$

$$\nabla f = \langle 3x^2 + yz,\ 3y^2 + xz,\ 3z^2 + xy \rangle$$

$$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$$

$$u = \frac{\langle -2, -2, -2 \rangle}{\sqrt{12}} = \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

$$\langle 4,4,4 \rangle \cdot \left\langle -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle = -\frac{12}{\sqrt{3}}$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos(2)$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= 6x \cdot e^u \cos v - 6y\, e^u \sin v$$

at $(u, v) = (0, 1)$, $(x, y) = (\cos(1), \sin(1))$

$$\frac{\partial g}{\partial u} = 6\cos^2(1) - 6\sin^2(1) = 6\cos(2)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 64π

Using divergence theorem:

$$\int_S F\cdot dS = \iiint_E div(F)\, dV$$

$$div(F) = 3 - 2 + 5 = 6$$

$$\iiint_E 6\, dV = \text{Volume of } E \cdot 6$$

Volume of E $= \frac{4}{3}\pi 2^3 = \frac{32\pi}{3}$

(Sphere radius 2)

$$\frac{32\pi}{3} \cdot 6 = 64\pi$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3z \,,\, 2x \,,\, y+z \,\rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

$$\iint_S F \cdot dS = \int_0^1 \int_0^1 \left(-P \frac{\partial g}{\partial x} - Q \frac{\partial g}{\partial y} + R\right) dy\,dx$$

$$= \int_0^1 \int_0^1 \left(-(12x+18y) - (6x) + 2x + 4y\right) dy\,dx$$

$$= \int_0^1 \int_0^1 (-16x - 14y)\, dy\,dx$$

With maple:

int(int(−16·x − 14·y, y=0..1), x=0..1);

returns −15

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. $\left(\frac{3}{4}, -1\right)$ saddle point

$$f_x = 4 - \frac{2}{2x+y} \qquad f_y = -2y - \frac{1}{2x+y}$$

Solving the system:

$$4 - \frac{2}{2x+y} = 0 \quad , \quad -2y - \frac{1}{2x+y} = 0$$

We get $x = \frac{3}{4}$ and $y = -1$

$$D = f_{xx} f_{yy} - f_{xy}^2 = \frac{4}{(2x+y)^2} \cdot \left(-2 + \frac{1}{(2x+y)^2}\right) - \left(\frac{2}{(2x+y)^2}\right)^2$$

At $x = \frac{3}{4}$, $y = -1$

$D = -32$, So it's a saddle point

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.000333

$$f(1,1,2) = 3$$

$$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} \overset{\text{at}(1,1,2)}{=} \frac{2}{3}$$

$$f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} \overset{\text{at}(1,1,2)}{=} 1$$

$$f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}} \overset{\text{at}(1,1,2)}{=} \frac{2}{3}$$

$$L(x,y,z) = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2)$$

$$L(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(0.001) - 0.001 + \frac{2}{3}(0.001)$$

$$= 3 + 0.001333 - 0.001 \approx 3.000333$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. $\frac{\sqrt{2}}{6}$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\sqrt{2}}{2\cos\theta}} r^2 \cos\theta\,dr\,d\theta$$

$$\int_0^{\frac{\sqrt{2}}{2\cos\theta}} r^2 \cos\theta\,dr = \frac{\left(\frac{\sqrt{2}}{2\cos\theta}\right)^3 \cos\theta}{3}$$

$$\frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\sqrt{2}}{4\cos^2\theta}\,d\theta = \frac{1}{3}\left(\frac{\sqrt{2}\tan\theta}{4}\right)_0^{\frac{\pi}{4}} = \frac{\sqrt{2}}{12}$$

$$\int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx = \int_0^{\frac{\pi}{4}} \int_{\frac{\sqrt{2}}{2\cos\theta}}^1 r^2\cos\theta\,dr\,d\theta$$

$$\int_{\frac{\sqrt{2}}{2\cos\theta}}^1 r^2\cos\theta\,dr = \frac{\cos\theta - \frac{\sqrt{2}}{4\cos^2\theta}}{3}$$

$$\frac{1}{3}\int_0^{\frac{\pi}{4}} \left(\cos\theta - \frac{\sqrt{2}}{4\cos^2\theta}\right) d\theta = \frac{1}{3}\left(\sin\theta - \frac{\sqrt{2}\tan\theta}{4}\right)_0^{\frac{\pi}{4}}$$

$$= \frac{1}{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{4}\right) = \frac{\sqrt{2}}{12}$$

$$\text{result} = \frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{12} = \frac{\sqrt{2}}{6}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\int_{-\frac{\pi}{2}}^{0} \int_0^2 \int_0^2 (r^4 \cos^2\theta \, \sin\theta \, z) \, dz\, dr\, d\theta$

$0 \leq \theta \leq -\frac{\pi}{2}$ $\quad 0 \leq r \leq 2$ $\quad 0 \leq z \leq 2$

$$\int_{-\frac{\pi}{2}}^{0} \int_0^2 \int_0^2 (r^4 \cos^2\theta \, \sin\theta \, z) \, dz\, dr\, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r'(\frac{\pi}{3}) = \langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$

$r''(\frac{\pi}{3}) = \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\rangle$

$$K(\tfrac{\pi}{3}) = \frac{|r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3})|}{|r'(\frac{\pi}{3})|^3} = \frac{9}{3^3} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 \,,\, uv \,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_u^1 (2u^2 + 2v^2)\, dv\, du$

$$\int_0^1 \int_u^1 |r_u \times r_v| \quad dv\, du$$

$$r_u = \langle 2u, v, 0 \rangle$$

$$r_v = \langle 0, u, 2v \rangle$$

$$r_u \times r_v = \langle 2v^2, -4uv, 2u^2 \rangle$$

$$|r_u \times r_v| = 2\sqrt{(u^2+v^2)^2} = 2u^2 + 2v^2$$

$$\int_0^1 \int_u^1 (2u^2 + 2v^2)\, dv\, du$$

16. (12 points) Let

$$f(x,y,z) \; = \; xy^2z^3 \quad ,$$

and let

$$g(x,y,z) \; = \; x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f)\,.\,grad(g) \qquad .$$

at the point $(1,1,1)$.

---

ans. 14

$grad(f) = \langle y^2z^3, \; 2xyz^3, \; 3xy^2z^2 \rangle$

at $(1,1,1) = \langle 1, \; 2, \; 3 \rangle$

$grad(g) = \langle 1, \; 2y, \; 3z^2 \rangle$

at $(1,1,1) = \langle 1, \; 2, \; 3 \rangle$

$grad(f) \cdot grad(g) = 1+4+9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

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ans. 0

Not sure how to prove it but plugging in values as it appraches from different directions seems to show the limit is 0.