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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18

2. $\int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$

3. $z = -x/2 - 5y/6 + 7\pi/18$

4. $13, 4, 11$

5. $\pi/3, \pi/3, \pi/3$

6. $-16/\sqrt{3}$

7. $6\cos^2(1) - 6\sin^2(1)$

8. 8π

9. -15

10. saddle point at $(3/4, -1)$

11. 3.06033

12. 0

13. $\int_0^2\int_0^{\pi/2}\int_{\pi/2}^{\pi} \rho^6 \sin^4\phi\cos\phi\cos^2\theta\sin\theta\,d\theta\,d\phi\,d\rho$

14. $1/3$

15. $\int_0^1\int_u^1 \sqrt{(2v^2)^2+(4uv)^2+(2u^2)^2}\,dv\,du$

16. 14

17. 0

Sign the following declaration:

I Angelica Armstrong Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Angelica Armstrong

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x,y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \xrightarrow{1} (-1,0) \xrightarrow{2} (-1,1) \xrightarrow{3} (0,2) \xrightarrow{4} (1,1) \xrightarrow{5} (1,0) \quad .$$

Explain!

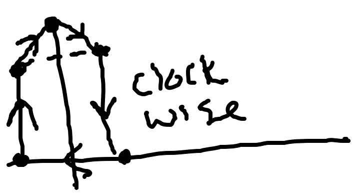
---

ans. -18

$dx\ dy = \partial D$ start = end = closed loop

Green's Theorem $\iint Q_x - P_y\, dA$

$\iint 11-5\, dA$ constant · area



rectangle $= 2 \cdot 1 = 2$

triangle $= \frac{1}{2} \cdot 2 \cdot 1 = 1$

$\Big\} 3$

$3 \cdot 6 = 18$

-18

.

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, dy\, dx \qquad .$$

---

ans. $\int_{\frac{1}{2}}^{1} \int_{y^2}^{1} f(x,y)\,dx\,dy$

$\frac{1}{4} < x < 1 \qquad 0 < y < \sqrt{x} \qquad y^2 < x$

$y^2 < x < 1 \qquad \frac{1}{2} < y < 1$

.

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) \;=\; 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

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ans. $z = -\frac{x}{2} - \frac{5y}{6} + \frac{7\pi}{18}$

$$f\left(\tfrac{\pi}{6}, \tfrac{\pi}{6}, \tfrac{\pi}{6}\right) = 2\cos\left(\tfrac{\pi}{3}\right) + 4\cos\left(\tfrac{\pi}{3}\right) + 8\cos\left(\tfrac{\pi}{3}\right) = 7$$

$$= 2 \cdot \tfrac{1}{2} + 4 \cdot \tfrac{1}{2} + 8 \cdot \tfrac{1}{2} = 7$$

$$= 1 + 2 + 4 = 7 \checkmark$$

$$f_x = -2\sin(x+y) - 4\sin(x+z)\Big|_{\frac{\pi}{6}} \to -2 \cdot \tfrac{\sqrt{3}}{2} - 4\tfrac{\sqrt{3}}{2}$$

$$f_y = -2\sin(x+y) - 8\sin(y+z)\Big|_{\frac{\pi}{6}} \to -2 \cdot \tfrac{\sqrt{3}}{2} - 8\tfrac{\sqrt{3}}{2}$$

$$f_z = -4\sin(x+z) - 8\sin(y+z)\Big|_{\frac{\pi}{6}} \to -4 \cdot \tfrac{\sqrt{3}}{2} - 8\tfrac{\sqrt{3}}{2}$$

$$0 = -3\sqrt{3}\left(x - \tfrac{\pi}{6}\right) - 5\sqrt{3}\left(y - \tfrac{\pi}{6}\right) - 6\sqrt{3}\left(z - \tfrac{\pi}{6}\right)$$

$$-3\sqrt{3}x + \frac{\sqrt{3}\pi}{2} - 5\sqrt{3}y + \frac{5\sqrt{3}\pi}{6} - 6\sqrt{3}z + \sqrt{3}\pi$$

$$6\sqrt{3}z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{8\sqrt{3}\pi}{6} + \sqrt{3}\pi$$

$$z = \left(-3\sqrt{3}x - 5\sqrt{3}y + \frac{14\sqrt{3}\pi}{6}\right) \Big/ 6\sqrt{3}$$

maple to simplify

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. 13, 4, 11

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distribute

$i, j, k = 1, 1, 1$

$-(a \times b) + (a \times 3c) + (b \times 2a) + (b \times 3c) + (c \times 2a) - (c \times b)$

$-\langle 1, 1, -1\rangle + \langle 6, 3, 6\rangle + \langle 2, 2, -2\rangle + \langle 3, -3, 3\rangle + \langle 4, 2, 4\rangle - \langle 1, -1, 1\rangle$

$i: -1 + 6 + 2 + 3 + 4 - 1 = 13$

$j: -1 + 3 + 2 - 3 + 2 + 1 = 4$

$k: 1 + 6 - 2 + 3 + 4 - 1 = 11$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: radians ; $\pi/3$

The angle at B is: radians ; $\pi/3$

The angle at C is: radians ; $\pi/3$

$AB = 1,0,1 \quad AC = 1,1,0 \quad BC = 0,1,-1 \quad \|AB\| = \|AC\| = \|BC\| = \sqrt{2}$

$\theta_A = \cos^{-1}\left(\frac{AB \cdot AC}{\|AB\| \cdot \|AC\|}\right) \qquad \cos^{-1}\left(\frac{1}{2}\right) = \pi/3$

$\theta_A = \theta_B = \theta_C$

b/c all are $\cos^{-1}\left(\frac{1}{2}\right)$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1,1,1)$ in a direction pointing to the point $(-1,-1,-1)$.

---

ans. $-16/\sqrt{3}$

$PQ = (-1-1, -1-1, -1-1) = (-2,-2,-2)$

$\sqrt{2^2+2^2+2^2} = \sqrt{12}$ unit vecter $\frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}$

$\nabla f = 3x^2+yz, 3y^2+xz, 3z^2+xy$

$\nabla f_{(1,1,1)} = 4, 4, 4$

$\langle 4,4,4 \rangle \cdot \langle \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}}, \frac{-2}{\sqrt{12}} \rangle$

$= \frac{-8}{\sqrt{12}} + \frac{-8}{\sqrt{12}} + \frac{-8}{\sqrt{12}} = \frac{-32}{\sqrt{12}}$

$\frac{-32}{2\sqrt{3}} = \frac{-16}{\sqrt{3}}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

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ans. $6\cos^2(1) - 6\sin^2(1)$

$$\frac{dg}{du} = \frac{dg}{dx} \cdot \frac{dx}{du} + \frac{dg}{dy} \cdot \frac{dy}{du}$$

$$= 6x \cdot e^u \cos v + -6y \cdot e^u \sin v$$

$$= 6(e^u \cos v) \cdot e^u \cos v - 6(e^u \sin v) \cdot e^u \sin v \Big|_{0,1}$$

$$= 6\cos(1) \cdot \cos(1) - 6\sin(1) \cdot \sin(1)$$

$$= 6\cos^2(1) - 6\sin^2(1)$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

Divergence Theorem

$div(F) = 3 - 2 + 5 = 6$

$6 \iiint_R 1 \, dV = 6 \cdot$ volume

its a sphere so volume $\rightarrow \frac{4}{3}\pi r^3$

$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \cdot 6$

$= 64\pi / 8$

for bands

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3z\,,\, 2x\,,\, y+z \,\rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15

$z = 2x+3y$ so use $\iint -P\frac{dg}{dx} - Q\frac{dg}{dy} + R$

$-3z(2) - 2x(3) + y + z$

$-6(2x+3y) - 6x + y + 2x + 3y$

$\longrightarrow -12x - 18y - 6x + y + 2x + 3y$

$\int_0^1\int_0^1 -16x - 14y$ ← maple

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. $(3/4, -1)$ saddle point

$f_x = 4 - \frac{2}{2x+y} = 4 - 2(2x+y)^{-1}$ $\qquad (3/4, -1)$

$f_y = -2y - \frac{1}{2x+y} = -2y - (2x+y)^{-1}$

$D = 16 \cdot 2 - (8)^2$
$= 32 - 64 = -32 < 0$

$f_{xx} = 2(2x+y)^{-2} \cdot 2 \to 16$

$f_{yy} = -2 + (2x+y)^{-2} \to 2$

$f_{xy} = 2(2x+y)^{-2} \to 8$

$3/4, -1$

$4 - \frac{2}{2x+y} = 0$ and $-2y - \frac{1}{2x+y} = 0$

$-4xy - 2y^2 = 1$

$8x + 4y = 2$

$x = \frac{2 - 4y}{8}$

$-\cancel{4}\left(\frac{2-4y}{\cancel{8}\,2}\right)y = \left(\frac{-2+4y}{2}\right)y$

$\frac{-2y + 4y^2}{2} = -y + 2y^2 - 2y^2 = 0$

$-y = 1$

$y = -1$

$8x - 4 = 2$

$8x = 6$

$x = 3/4$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.00033

$f = (2x^2+3y^2+z^2)^{1/2} \to (2+3+4)^{1/2} = 3$

$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 4x \to 2(1)(2+3+4)^{-1/2} = \frac{2}{3}$

$f_y = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 6y \to 3(1)(2+3+4)^{-1/2} = \frac{3}{3}$

$f_z = \frac{1}{2}(2x^2+3y^2+z^2)^{-1/2} \cdot 2z \to 2(2+3+4)^{-1/2} = \frac{2}{3}$

$1,1,2$

$L = 3 + \frac{2}{3}(x-1) + 1(y-1) + \frac{2}{3}(z-2) \Big|_{\substack{1.001\\0.999\\2.001}}$

$L = 3 + \frac{2}{3}(.001) - (.001) + \frac{2}{3}(.001)$

$= 3 + (.000667) - (.001) + (.000667) = 3.00033$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

---

ans. 0

$r=1$ $\quad x = \cos(t)$

closed loop

13. (12 points) Convert the triple iterated integral

$$\int_0^2 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^0 x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

---

ans. $\int_0^2 \rho^6\, d\rho \cdot \int_0^{\pi/2} \sin^4\phi \cdot \cos\phi\, d\phi \cdot \int_{\pi/2}^{\pi} \cos^2\theta \cdot \sin\theta\, d\theta$

$0<z<2 \quad 0<y<\sqrt{4-z^2} \quad -\sqrt{4-z^2-y^2}<x<0$

$x<0,\ y>0,\ z>0 \quad 0<\rho<2,\ \pi/2<\theta<\pi,\ 0<\phi<\pi/2$

$(\rho\sin\phi\cos\theta)^2 \cdot \rho\sin\phi\sin\theta \cdot \rho\cos\phi \cdot \rho^2\sin\phi$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $1/3$

$$\frac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$$

$r' = 0, 3\cos t, -3\sin t \;\to 0, 3/2, -3\sqrt{3}/2$

$r'' = 0, -3\sin t, -3\cos t \;\to 0, -3\sqrt{3}/2, -3/2$

$\pi/3$

$$\begin{matrix} 0 & \frac{3}{2} & \frac{-3\sqrt{3}}{2} \\ 0 & \frac{-3\sqrt{3}}{2} & \frac{-3}{2} \end{matrix}$$

$= -9, 0, 0$

$$\frac{\|r' \times r''\|}{\|r'\|} = \frac{9}{3^3} = \frac{9}{27} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) \;=\; \langle u^2\,,\, uv\,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_u^1 \sqrt{(2v^2)^2+(4uv)^2+(2u^2)^2}\; dv\, du$

$r = u^2 + uv + v^2$ $\qquad$ $u = x \quad v = y$

$r_u = 2u, v, 0 \qquad r_v = 0, u, 2v$

$$\begin{matrix} 2u & v & 0 \\ 0 & u & 2v \end{matrix} = 2v^2, -4uv, 2u^2$$

$\sqrt{(2v^2)^2 + (4uv)^2 + (2u^2)^2}$

$\int_0^1 \int_v^1$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \,.\, grad(g) \qquad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$grad f = y^2z^3, x2yz^3, xy^2 3z^2 |_{1,1,1} = 1, 2, 3$

$grad g = 1, 2y, 3z^2 |_{1,1,1} = 1, 2, 3$

$\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1 + 4 + 9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

---

ans. 0

$y-b=c(x-a)$

$y=cx$

$$\frac{(x+cx)^2-(z+w)^2}{x+cx-z-w}$$

maple

$w-e=c(x-a)$

$w=cx$

$z-d=c(x-a)$

$z=cx$

$$\lim_{x\to 0} \frac{(x+cx)^2-(cx+cx)^2}{x+cx-cx-cx}$$

$$\lim_{x\to 0} \frac{x^2+2cx^2+c^2x^2-4c^2x^2}{x-cx}$$

take out $\frac{x^2}{x}$

still depends on x

Just plug in close values

like .0001