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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 3
2. $\int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dx\,dy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dx\,dy$
3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$
4. $\langle 3,-6,9\rangle$
5. $\pi/3,\ \pi/3,\ \pi/3$
6. $-4\sqrt{3}$
7. $g_u(0,1) = 6\cos(2)$
8. 8π
9. -15
10. $(\frac{3}{4},-1)$ is a saddle point
11. $\frac{9001}{3000}$
12. $\frac{1}{3\sqrt{2}}$
13. $\int_0^{2\pi}\int_0^{\pi/2}\int_{\pi/2}^{\pi} \rho^6 \sin^4\phi\cos^2\theta\cos\phi\sin\theta\,d\rho\,d\phi\,d\theta$
14. $\frac{1}{3}$
15. $\int_0^1\int_0^v 2v\sqrt{v^2+4u^2}\,du\,dv$
16. 14
17. 0

Sign the following declaration:

I   Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx + (sin\,(e^{\cos y}) + 11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans. 3

$\leftarrow \frac{1}{2}bh$ $\quad b=2$, $h=1$ $\quad =1$

$L \cdot W = 2 \cdot 1 = 2$

Area $= 2+1 = 3$

No time

Greene's theorem?

$\int_C P\,dx + Q\,dy$

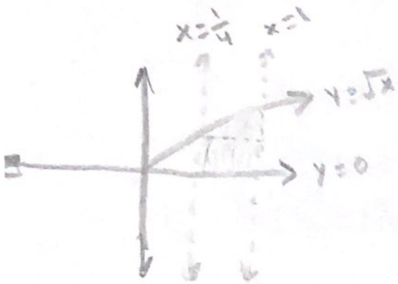
2. (12 points) Change the order of integration

dydx

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x,y)\, \cancel{dx\, dy} \quad .$$

ans.

$$\int_0^{1/2}\int_{1/4}^{1} f(x,y)\,dxdy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)\,dxdy$$



Splitting into 2 parts, does not work with 1

$$\int_0^{1/2}\int_{1/4}^{1} f(x,y)dxdy + \int_{1/2}^{1}\int_{y^2}^{1} f(x,y)dxdy$$

add

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$.

$f(x,y,z) = 2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7$

$f_x = -4\sin(x+z) - 2\sin(x+y)$

$f_y = -8\sin(y+z) - 2\sin(x+y)$

$f_z = -8\sin(y+z) - 4\sin(x+z)$

$\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6} = -4\sin(\frac{\pi}{3}) - 2\sin(\frac{\pi}{3}) = -4\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} = \boxed{-3\sqrt{3}}$

$= -8\sin(\frac{\pi}{3}) - 2\sin(\frac{\pi}{3}) = -8\cdot\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} = \boxed{-5\sqrt{3}}$

$= -8\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3}) = -8\cdot\frac{\sqrt{3}}{2} - 4\cdot\frac{\sqrt{3}}{2} = \boxed{-6\sqrt{3}}$

$-3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6}) = 0$

$-3x - 5y - 6z + \frac{7\pi}{3} = 0$

$\pi + \frac{5\pi}{6} + \frac{3\pi}{6}$

$\frac{6\pi}{6} + \frac{5\pi}{6} + \frac{3\pi}{6} = \frac{14\pi}{6} = \frac{7\pi}{3}$

$\frac{6z}{6} = \frac{-3x - 5y + \frac{7\pi}{3}}{6}$

$z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{18}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

ans. $\langle 3, -6, 9 \rangle$

$a \times (2a - b + 3c) + b \times (2a - b + 3c) + c \times (2a - b + 3c)$

$\cancel{a \times 2a}^{0} - a \times b + a \times 3c + b \times 2a - \cancel{b \times b}^{0} + b \times 3c + c \times 2a - c \times b + \cancel{c \times 3c}^{0}$

$-a \times b + a \times 3c + b \times 2a + b \times 3c + c \times 2a - c \times b$

$\langle -1, -1, -1 \rangle + \langle 6, 3, 6 \rangle + \langle -2, -2, 2 \rangle + \langle 3, -3, 3 \rangle + \langle -4, -2, -4 \rangle + \langle 1, -1, 1 \rangle$

$\langle 5, 2, 7 \rangle + \langle 1, -5, 5 \rangle + \langle -3, -3, -3 \rangle$

$\langle 6, -3, 12 \rangle + \langle -3, -3, -3 \rangle = \langle 3, -6, 9 \rangle$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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ans. The angle at A is: $\pi/3$ radians ;

The angle at B is: $\pi/3$ radians ;

The angle at C is: $\pi/3$ radians ;

$$\cos\theta = \frac{u \cdot v}{|u| \cdot |v|}$$

$\overline{AB} = (1,0,1) - (0,0,0) = \langle 1,0,1 \rangle$

$\overline{AC} = (1,1,0) - (0,0,0) = \langle 1,1,0 \rangle$

$\overline{BC} = (1,1,0) - (1,0,1) = \langle 0,1,-1 \rangle$

$\overline{BA} = (0,0,0) - (1,0,1) = \langle -1,0,-1 \rangle$

$$\cos\theta_A = \frac{\overline{AB} \cdot \overline{AC}}{|\overline{AB}| \cdot |\overline{AC}|} = \frac{(1+0+0)}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

$$\theta_A = \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3$$

$$\theta_B = \cos^{-1}\left(\tfrac{1}{2}\right) = \pi/3$$

$$\cos\theta_B = \frac{(0+0+1)}{\sqrt{2} \cdot \sqrt{2}} = \frac{1}{2}$$

$$\theta_C = \pi - \theta_A - \theta_B = \pi/3$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $\overset{P}{(1, 1, 1)}$ in a direction pointing to the point $\overset{Q}{(-1, -1, -1)}$.

---

ans. $-4\sqrt{3}$

$\nabla f = \langle 3x^2+yz,\ 3y^2+xz,\ 3z^2+xy \rangle$

$Q-P = \langle -2, -2, -2 \rangle \qquad |Q-P| = \sqrt{12} = 2\sqrt{3}$

$u = \dfrac{\swarrow}{2\sqrt{3}} = \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$

$\nabla f(1,1,1) = \langle 4, 4, 4 \rangle$

$\nabla f(1,1,1) \cdot u \qquad \langle 4,4,4 \rangle \cdot \left\langle \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}} \right\rangle$

$= \frac{-4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} = \frac{-12}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{-12}{3} \cdot \sqrt{3} = \boxed{-4\sqrt{3}}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $g_u(0,1) = 6\cos 2$

$g_u = (g_x)(x_u) + (g_y)(y_u)$ $\quad g_x = 6x$, $g_y = -6y$, $g_x = 6e^u \cos v$, $g_y = -6e^u \sin v$

$x_u = e^u \cos v \quad y_u = e^u \sin v$

$g_u = (6e^u \cos v)(e^u \cos v) - (6e^u \sin v)(e^u \sin v)$

$u \quad v$

$g_u(0,1) = (6\cos(1))(\cos(1)) - (6\sin(1))(\sin(1))$

$= 6\cos^2(1) - 6\sin^2(1)$

$g_u(0,1) = 6(\cos^2(1) - \sin^2(1))$ ← double angle $\left(\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)\right)$

$= 6\cos(2)$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 8π

$x^2+y^2+z^2<4$

$\iint F\cdot dS = \iiint_D \operatorname{div} F\,dV$. $\operatorname{div} F = 3-2+5$

$= 6$

$\iiint_D 6\,dV$

$6\cdot\left(\frac{4}{3}\pi r^3\right)$ $r=2$ (1/8 of a sphere)

$6\cdot\frac{4}{3}\pi r^3 = 8\pi r^3 = 8\pi\cdot 8 = \frac{64\pi}{8} = 8\pi$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

P Q R

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

$g(x,y) = 2x + 3y \qquad g_x = 2 \qquad g_y = 3 \qquad \iint_S F \cdot ds = \iint_D \left(-P\frac{dz}{dx} - Q\frac{dz}{dy} + R\right) dA$

$$\int_0^1\int_0^1 (-6z - 6x + y + z)\,dx\,dy$$

$$\int_0^1\int_0^1 (-12x - 6y - 6x + y + 2x + 3y)\,dx\,dy$$

$$\int_0^1\int_0^1 (-16x - 14y)\,dx\,dy = -15$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

ans. $(\frac{3}{4}, -1)$ is a saddle point

$$f_x = 4 - \frac{2}{2x+y} \qquad f_y = -2y - \frac{1}{2x+y}$$

$$f_x = 0 \rightarrow 4 - \frac{2}{2x+y} = 0 \qquad 8x + 4y = 2$$

$$f_y = 0 \rightarrow -2y - \frac{1}{2x+y} = 0 \rightarrow 2y = \frac{-1}{2x+y} \rightarrow 4xy + 2y^2 = -1$$

$$cp: (\tfrac{3}{4}, -1) \qquad x = \tfrac{3}{4}, \ y = -1$$

$$f_{xx} = \frac{4}{(2x+y)^2} \Big|_{(\frac{3}{4},-1)} = 16$$

$$D = f_{xx} \cdot f_{yy} - (f_{xy})^2$$

$$f_{xy} = \frac{2}{(2x+y)^2} \Big|_{(\frac{3}{4},-1)} = 8 \qquad D = (16)(2) - (64) = -32$$

$$f_{yy} = -2 + \frac{1}{(2x+y)^2} \Big|_{(\frac{3}{4},-1)} = 2$$

$$-32 < 0$$

$(\frac{3}{4}, -1)$ = saddle point

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

a b c

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

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ans. 9001/3000

$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$ $f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}}$ $f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$

$f_x(1,1,2) = \frac{2}{3}$ $f_y(1,1,2) = 1$ $f_z(1,1,2) = 2/3$

$L(x,y,z) = f(a,b,c) + f_x(x-a) + f_y(y-b) + f_z(z-c)$

$= 3 + 2/3(1.001-1) + 1(0.999-1) + 2/3(2.001-2)$

$= \frac{9001}{3000} = 3.00033$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad .$$

Explain!

ans. $\frac{1}{3\sqrt{2}}$

$0 \le r \le \frac{\sqrt{2}}{2\cos\theta}$ $\quad 0 \le \theta \le \pi/4$

$$\int_0^{\pi/4} \int_0^{\frac{\sqrt{2}}{2\cos\theta}} r\cos\theta\, r\, dr\, d\theta$$

$\frac{\sqrt{2}}{2\cos\theta} \le r \le 1 \quad 0 \le \theta \le \frac{\pi}{4}$

$$\int_0^{\pi/4} \int_{\frac{\sqrt{2}}{2\cos\theta}}^{1} r\cos\theta\, r\, dr\, d\theta$$

$$\int_0^{\pi/4} \int_0^{\frac{\sqrt{2}}{2\cos\theta}} r\cos\theta\, r\, dr\, d\theta + \int_0^{\pi/4} \int_{\frac{\sqrt{2}}{2\cos\theta}}^{1} r\cos\theta\, r\, dr\, d\theta = \frac{1}{3\sqrt{2}}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^{\cancel{4}\,2} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz \qquad d\rho\ d\theta\ d\rho$$

to **spherical coordinates. Do not evaluate.**

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ans. $\int_0^2 \int_0^{\pi/2} \int_{\pi/2}^{\pi} \rho^6 \sin^4\varphi \cos^2\theta \cos\varphi \sin\theta \, d\rho\, d\varphi\, d\theta$

$x^2+y^2+z=4$

$0 \le \varphi \le \pi/2$

$0 \le \rho \le 2$

$\frac{\pi}{2} \le \theta \le \pi$

$$\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^{2} (\rho^2 \sin^2\theta \cos^2\theta)(\rho \sin^2\theta)(\rho\cos\theta)\rho^2 \sin\theta \, d\rho\, d\varphi\, d\theta$$

$$= \rho^6 \sin^4\varphi \cos^2\theta \cos\varphi \sin\theta \, d\rho\, d\varphi\, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $\frac{1}{3}$

$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$, $r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$r'(\pi/3) = \langle 0, 3/2, \frac{-3\sqrt{3}}{2} \rangle$, $r''(\pi/3) = \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2} \rangle$

$r'(\pi/3) \times r''(\pi/3) = \langle -9, 0, 0 \rangle$ $\quad |\langle -9, 0, 0 \rangle| = 9$

$|r'(\pi/3)| = \sqrt{0 + \frac{9}{4} + \frac{27}{4}} = \sqrt{\frac{36}{4}} = \sqrt{9} = 3$

$$K(\tfrac{\pi}{3}) = \frac{|r'(\frac{\pi}{3}) \times r''(\frac{\pi}{3})|}{|r'(\frac{\pi}{3})|^3} = \frac{9}{3^3} = \boxed{\frac{1}{3}}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u, v) = \langle u^2 \,,\, uv \,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

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ans. $\int_0^1 \int_0^v 2v\sqrt{v^2+4u^2}\, du\, dv$

$\iint_D \|N(u,v)\|\, du\, dv$

$N(u) = 2u, v, 0$

$N(v) = 0, u, 2v$

$N(u) \times N(v)$

$2v^2 i - 4uv j \quad = \langle 2u^2, -4uv, 0 \rangle$

$|\langle 2u^2, -4uv, 0 \rangle| = 2v\sqrt{v^2+4u^2}$

$\int_0^1 \int_0^v 2v\sqrt{v^2+4u^2}\, du\, dv$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1, 1, 1)$.

---

ans. 14

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$\nabla f(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle \;, \; \nabla g(1,1,1) = \langle 1,2,3 \rangle$

$\nabla f(1,1,1) \cdot \nabla g(1,1,1) = \langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle = 1+4+9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. 0

$$=\frac{0}{0}$$

Simplified:

difference of squares

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{\cancel{(x+y-z-w)}(x+y+z+w)}{\cancel{x+y-z-w}}$$

$$\lim_{(x,y,z,w)\to(0,0,0,0)} (x+y+z+w) = 0$$

limit exists, is equal to 0