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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

1.

2. $\int_1^{1/2}\int_0^{y^2} f(x,y)\,dx\,dy$

3. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{12}$

4. $\langle 3,-6,9 \rangle$

5. $\frac{\pi}{3}$; $\frac{\pi}{3}$; $\frac{\pi}{3}$

6. $-4\sqrt{3}$

7. $6\cos 2$

8.

9.

10. LOCAL MIN @ $(-\frac{1}{4}, 1)$

11. 8.001

12.

13.

14. $1/3$

15. $\iint_D |\langle 2v^2, -4uv, 2u^2 \rangle|\,du\,dv$, $D = \{(u,v) \mid 0 < u < v < 1\}$

16. 14

17. it exists and is $\emptyset$

Sign the following declaration:

I   Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed:

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

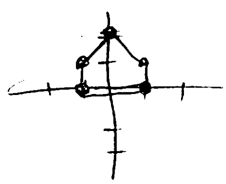
over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$
$$①\qquad ②\qquad ③\qquad ④\qquad ⑤$$

Explain!

---

ans.



① $V = \langle 1,0\rangle + t\langle -2,0\rangle = \langle 1,0\rangle + \langle -2t,0\rangle = \langle 1-2t,0\rangle$

$\int_0^1 \cos e^{\sin(1-2t)} + \sin e^{1}\; 11\,(1-2t) \cdot dt$

② $V = \langle -1,0\rangle + t\langle 0,1\rangle = \langle -1,0\rangle + \langle 0,t\rangle = \langle -1,t\rangle$

$\int_0^1 \cos e^{\sin -1} + 5t + \sin e^{\cos t} - 11\, dt$

③ $V = \langle -1,1\rangle + t\langle 1,1\rangle = \langle -1,1\rangle + \langle t,t\rangle = \langle t-1,t+1\rangle$

$\int_0^1 \cos e^{\sin(t-1)} + 5(t+1) + \sin e^{\cos(t+1)} + 11(t-1)\, dt$

④ $V = \langle 0,2\rangle + t\langle 1,-1\rangle = \langle 0,2\rangle + \langle t,-t\rangle = \langle t,2-t\rangle$

$\int_0^1 \cos e^{\sin t} + 5(2-t) + \sin e^{\cos(2-t)} + 11t\, dt$

⑤ $V = \langle 1,1\rangle + t\langle 0,-1\rangle = \langle 1,1\rangle + \langle 0,-t\rangle = \langle 1,1-t\rangle$

$\int_0^1 \cos e + 5(1-t) + \sin e^{\cos(1-t)} + 11$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1} \int_{0}^{\sqrt{x}} f(x, y) \, \cancel{dx\, dy} \; dy\, dx \quad .$$

---

ans. $\int_{1/2}^{1} \int_{0}^{y^2} f(x,y)\, dx\, dy$

$0 \le y \le \sqrt{x}$ $\qquad \sqrt{\frac{1}{4}} = \frac{1}{2}$

$\frac{1}{4} \le x \le 1$ $\qquad \sqrt{1} = 1$

$y = \sqrt{x}$

$x = y^2$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z = -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{12}$

$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$

$F_x(x - x_0) + F_y(y - y_0) + F_z(z - z_0) = 0$

$F_x = -2\sin(x+y) - 4\sin(x+z) \Rightarrow$ @ point $= -2\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3})$

$F_y = -2\sin(x+y) - 8\sin(y+z) \qquad = -2\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})$

$F_z = -4\sin(x+z) - 8\sin(y+z) \qquad = -4\sin(\frac{\pi}{3}) - 8\sin(\frac{\pi}{3})$

$F_x = -2\frac{\sqrt{3}}{2} - 4\frac{\sqrt{3}}{2} = -\sqrt{3} - 2\sqrt{3} = -3\sqrt{3}$

$F_y = -2\frac{\sqrt{3}}{2} - 8\frac{\sqrt{3}}{2} = -\sqrt{3} - 4\sqrt{3} = -5\sqrt{3}$

$F_z = -4\frac{\sqrt{3}}{2} - 8\frac{\sqrt{3}}{2} = -2\sqrt{3} - 4\sqrt{3} = -6\sqrt{3}$

$-3\sqrt{3}(x - \frac{\pi}{6}) - 5\sqrt{3}(y - \frac{\pi}{6}) - 6\sqrt{3}(z - \frac{\pi}{6}) = 0$

$-3\sqrt{3}x + \frac{3\pi\sqrt{3}}{6} - 5\sqrt{3}y + \frac{5\sqrt{3}\pi}{6} - 6\sqrt{3}z + \frac{6\sqrt{3}\pi}{6} = 0$

$6\sqrt{3}z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{14\sqrt{3}\pi}{6}$

$z = \frac{1}{6\sqrt{3}}\left(-3\sqrt{3}x - 5\sqrt{3}y + \frac{7\sqrt{3}\pi}{2}\right.$

$= -\frac{3\sqrt{3}}{6\sqrt{3}}x - \frac{5\sqrt{3}}{6\sqrt{3}}y + \frac{7\sqrt{3}\pi}{12\sqrt{3}}$

$= -\frac{1}{2}x - \frac{5}{6}y + \frac{7\pi}{12}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\overset{\langle 1,1,-1\rangle}{\mathbf{a}\times\mathbf{b} = \mathbf{i}+\mathbf{j}-\mathbf{k}} \quad , \quad \overset{\langle 1,-1,1\rangle}{\mathbf{b}\times\mathbf{c} = \mathbf{i}-\mathbf{j}+\mathbf{k}} \quad , \quad \overset{\langle 2,1,2\rangle}{\mathbf{a}\times\mathbf{c} = 2\mathbf{i}+\mathbf{j}+2\mathbf{k}} \quad .$$

What is

$$(\mathbf{a}+\mathbf{b}+\mathbf{c})\times(2\mathbf{a}-\mathbf{b}+3\mathbf{c}) \qquad ?$$

---

ans.

$$(a\times 2a) + (a\times -b) + (a\times 3c) + (b\times 2a) + (b\times -b) + (b\times 3c)$$
$$+ (c\times 2a) + (c\times -b) + (c\times 3c)$$

$$= \cancel{2(a\times a)} - (a\times b) + 3(a\times c) + 2(b\times a) \cancel{- (b\times b)} + 3(b\times c)$$
$$+ 2(c\times a) - (c\times b) + \cancel{3(c\times c)}$$

$$= -(a\times b) + 3(a\times c) + \underbrace{2(b\times a)}_{} + 3(b\times c) + \overset{-2(a\times c)+(b\times c)}{2(c\times a) - (c\times b)}$$

$-2(a\times b)$

$$= -\langle 1,1,-1\rangle + 3\langle 2,1,2\rangle + \_\_\_\_ + 3\langle 1,-1,1\rangle + \_\_\_ + \_\_\_$$

$$= -\langle 1,1,-1\rangle + \langle 6,3,6\rangle - 2\langle 1,1,-1\rangle + \langle 3,-3,3\rangle - 2\langle 2,1,2\rangle + \langle 1,-1,1\rangle$$

$\langle -1,-1,1\rangle$
$\langle 6,3,6\rangle$
$\langle -2,-2,2\rangle$
$\langle 3,-3,3\rangle$
$\langle -4,-2,-4\rangle$
$\langle 1,-1,1\rangle$

$\langle 3,-6,9\rangle$

5. (12 points) Find the three angles of the triangle ABC where

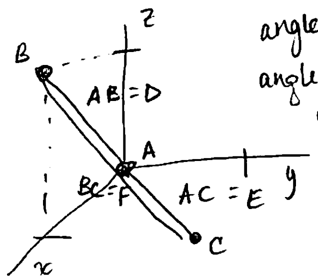
$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: $\frac{\pi}{3}$ radians ;

The angle at B is: $\frac{\pi}{3}$ radians ;

The angle at C is: $\frac{\pi}{3}$ radians ;



angle @ A = D × E
angle @ B = D × F
@ C = E × F

$|A \times B| = |A||B| \sin\theta$

$a_2b_3 - b_2a_3,\ b_1a_3 - a_1b_3,$
$a_1b_2 - b_1a_2$

$\vec{D} = \overline{AB} = B - A = (1,0,1) - (0,0,0) = \langle 1,0,1\rangle$

$\vec{E} = \overline{AC} = C - A = C = \langle 1,1,0\rangle$

$\vec{F} = \overline{BC} = C - B = (1,1,0) - (1,0,1) = \langle 0,1,-1\rangle$

@A: $|D \times E| = |D||E| \sin\theta$

$|\langle 1,0,1\rangle \times \langle 1,1,0\rangle| = |\langle 1,0,1\rangle||\langle 1,1,0\rangle| \sin\theta$

$|\langle 0-1, 1-0, 1-0\rangle| = \sqrt{1^2+1^2}\sqrt{1^2+1^2} \sin\theta$

$|\langle -1,1,1\rangle| = \sqrt{2}\sqrt{2} \sin\theta$

$\sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

@B: $|D \times F| = |D||F| \sin\theta$

$|\langle 1,0,1\rangle \times \langle 0,1,-1\rangle| = 2\sin\theta$

$|\langle 0-1, 0+1, 1-0\rangle| = 2\sin\theta$

$\sin\theta = \frac{|\langle -1,1,1\rangle|}{2} = \frac{\sqrt{3}}{2} \Rightarrow \frac{\pi}{3}$

@C: $|E \times F| = |E||F| \sin\theta$

$|\langle 1,1,0\rangle \times \langle 0,1,-1\rangle| = 2\sin\theta$

$|\langle -1-0, 0+1, 1-0\rangle| = 2\sin\theta$

$\sin\theta = \frac{|\langle -1,1,1\rangle|}{2} = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}$

6. (12 points) Find the directional derivative of

$$f(x,y,z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $\overset{P}{(1,1,1)}$ in a direction pointing to the point $\overset{Q}{(-1,-1,-1)}$.

---

ans. $-4\sqrt{3}$

$f_x = 3x^2 + yz$, $f_y = 3y^2 + xz$, $f_z = 3z^2 + xy$

$\nabla f(1,1,1) = \langle 3+1, 3+1, 3+1 \rangle = \langle 4,4,4 \rangle$

direction $= (-1,-1,-1) - (1,1,1) = \langle -2,-2,-2 \rangle$

$|\langle -2,-2,-2 \rangle| = \sqrt{3(2^2)} = \sqrt{3(4)} = 2\sqrt{3} \Rightarrow \frac{1}{2\sqrt{3}}$

$u = 2\sqrt{3}\,d = \langle -\frac{2}{2\sqrt{3}}, -\frac{2}{2\sqrt{3}}, -\frac{2}{2\sqrt{3}} \rangle = \langle -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \rangle$

$\nabla f \cdot u = \langle 4,4,4 \rangle \cdot \langle -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \rangle = -\frac{4\sqrt{3}}{3} - \frac{4\sqrt{3}}{3} - \frac{4\sqrt{3}}{3} = -\frac{12\sqrt{3}}{3} = -4\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos 2$

$$\frac{dg}{du} = \frac{dg}{dx}\frac{dx}{du} + \frac{dg}{dy}\frac{dy}{du}$$

$$= (6x)(e^u \cos v) + (-6y)(e^u \sin v)$$

$$= 6e^u x \cos v - 6e^u y \sin v$$

$$= 6e^u (x \cos v - y \sin v)$$

$$= 6e^0 (\cos 1 \cos 1 - \sin 1 \sin 1)$$

$$= 6(\cos^2 1 - \sin^2 1) = 6\cos 2$$

$x = e^0 \cos 1 = \cos 1$

$y = e^0 \sin 1 = \sin 1$

$\frac{dx}{du} = e^u \cos v$

$\frac{dy}{du} = e^u \sin v$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\, ,\, -2y + e^{x+z^2}\, ,\, 5z + \sin(xy^3 + e^x)\, \rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans.

---

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans.

---

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. local min @ $(-\frac{1}{4}, 1)$

$$f_x = 4 - \frac{2}{2x+y} \qquad f_y = 2y - \frac{1}{2x+y} \qquad f_{xy} = \frac{2}{(2x+y)^2}$$

$$f_{xx} = \frac{4}{(2x+y)^2} \qquad f_{yy} = 2 + \frac{1}{(2x+y)^2}$$

$$0 = 4 - \frac{2}{2x+y} \qquad 0 = 2y - \frac{1}{2x+y}$$

$$4 = \frac{2}{2x+y} \qquad 2y = \frac{1}{2x+y} = 2 \Rightarrow y = 1$$

CRITICAL POINT $(-\frac{1}{4}, 1)$

$$2 = \frac{1}{2x+y}$$

$$2 = \frac{1}{2x+1} \Rightarrow 4x+2 = 1 \Rightarrow 4x = -1 \Rightarrow x = -\frac{1}{4}$$

$$f_{xx}(-\tfrac{1}{4},1) = \frac{4}{(-2(\frac{1}{4})+1)^2} = \frac{4}{(-\frac{1}{2}+1)^2} = \frac{4}{(\frac{1}{2})^2} = \frac{4}{\frac{1}{4}} = 16$$

$$f_{yy} = 2 + \frac{1}{(2(-\frac{1}{4})+1)^2} = 2 + \frac{1}{(-\frac{1}{2}+1)^2} = 2 + \frac{1}{(\frac{1}{2})^2} = 2 + \frac{1}{\frac{1}{4}} = 2+4 = 6$$

$$f_{xy} = \frac{2}{(2(-\frac{1}{4})+1)^2} = \frac{2}{(-\frac{1}{2}+1)^2} = \frac{2}{(\frac{1}{2})^2} = \frac{2}{\frac{1}{4}} = 8$$

$$D = f_{xx} f_{yy} - [f_{xy}]^2 = 16 \cdot 6 - 64 = 32$$

$D > 0$ AND $f_{xx} > 0 \Rightarrow$ LOCAL MIN

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

$$(1,1,2) = 3$$

---

ans. 3.001

$$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} \qquad f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} \qquad f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

$$f(1,1,2) = \sqrt{2+3+4} = \sqrt{9} = 3$$

$$f_x(1,1,2) = \frac{2}{3}, \quad f_y = \frac{3}{3} = 1, \quad f_z = \frac{4}{3}$$

$$\text{approx} = 3 + \frac{2}{3}(0.001) - 0.001 + \frac{4}{3}(0.001) = 3.001$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

①   ②

Explain!

---

ans.

---

① $D = \{ (x,y) \mid 0 \le y \le x ,\ 0 \le x \le \frac{\sqrt{2}}{2} \}$

$= \{ (r, \theta) \mid$

13. (12 points) Convert the triple iterated integral

$$\int_0^{\overset{2}{\cancel{4}}} \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z \, dx\, dy\, dz$$

to **spherical coordinates. Do not evaluate.**

---

ans.

---

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. $\frac{1}{3}$

$$K = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$

$r'(\frac{\pi}{3}) = \langle 0, 3\cos\frac{\pi}{3}, -3\sin\frac{\pi}{3}\rangle$

$= \langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle$

$r''(\frac{\pi}{3}) = \langle 0, -3\sin\frac{\pi}{3}, -3\cos\frac{\pi}{3}\rangle$

$= \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\rangle$

$r'(t) \times r''(t) = \langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle \times \langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\rangle$

$= \langle \frac{3}{2}(-\frac{3}{2}) - (-\frac{3\sqrt{3}}{2})(-\frac{3\sqrt{3}}{2}), 0-0, 0\rangle$

$= \langle -\frac{9}{4} - \frac{27}{4}, 0, 0\rangle = \langle -\frac{36}{4}, 0, 0\rangle = \langle -9, 0, 0\rangle$

$|r'(t)|^3 = |\langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\rangle|^3 = \left((\frac{3}{2})^2 + (\frac{3\sqrt{3}}{2})^2\right)^{3/2} = \left(\frac{9}{4} + \frac{27}{4}\right)^{3/2} = (9)^{3/2} = 3^3 = 27$

$|r'(t) \times r''(t)| = |\langle 9, 0, 0\rangle| = \sqrt{9^2} = 9$

$K = \frac{9}{27} = \frac{1}{3}$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 , uv , v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\iint_D |\langle 2v^2, -4uv, 2u^2 \rangle| \, du \, dv , \quad D = \{(u,v) \mid 0 < u < v < 1\}$

$$\int_a^b \int_c^d |r_u \times r_v| \, du \, dv$$

$r_u = \langle 2u, v, 0 \rangle$

$r_v = \langle 0, u, 2v \rangle$

$$\begin{aligned} r_u \times r_v &= \langle 2u, v, 0 \rangle \times \langle 0, u, 2v \rangle \\ &= \langle 2v^2 - 0, 0 - 4uv, 2u^2 - 0 \rangle \\ &= \langle 2v^2, -4uv, 2u^2 \rangle \end{aligned}$$

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14

$$\nabla f = \langle f_x, f_y, f_z \rangle = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$$

$$\nabla g = \langle g_x, g_y, g_z \rangle = \langle 1, 2y, 3z^2 \rangle$$

$$\nabla f \cdot \nabla g = y^2z^3 + 4xy^2z^3 + 9xy^2z^5$$

$$= 1 + 4 + 9 = 14$$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans.

→ plugging into original equation we get $\frac{0}{0}$, try proving it does not exist if approaching from different directions

$\lim_{x\to 0} \Rightarrow \frac{x^2}{x} = x \Rightarrow 0$

$\lim_{y\to 0} \Rightarrow \frac{y^2}{y} = y \Rightarrow 0$

$\lim_{z\to 0} \Rightarrow \frac{-z^2}{-z} = z \Rightarrow 0$

$\lim_{w\to 0} \Rightarrow \frac{-w^2}{-w} = w \Rightarrow 0$

→ when approaching from different directions, the limit is the same