NAME: (print!) Aditya Sivakumar RUID: (print!) 201001195

SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. -18
2. $\int_0^1 \int_0^{y^2+12} f(x,y)\,dx\,dy$
3. $z = 7\pi/18 - \frac{5}{6}y - \frac{1}{2}x$
4. $\langle 0,0,10 \rangle$
5. Angle at A = Angle at B = Angle at C = $\pi/3$ radians
6. $\langle -4/\sqrt{3}, -4/\sqrt{3}, -4/\sqrt{3} \rangle$
7. $6\cos(2)$
8. 8π
9. -15
10. $(-3/4, 1)$; Saddle Point
11. $\frac{9001}{3000}$
12. $\longrightarrow \sqrt{2}/6$
13. $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^2 \rho^6 \sin^4\varphi \cos\varphi \sin\theta \cos^2\theta \, d\rho\, d\theta\, d\varphi$
14. $1/3$
15. $\int_0^1 \int_u^1 2\sqrt{v^4 + 4u^2v^2 + u^4}\, dv\, du$
16. 14
17. DNE

Sign the following declaration:

I Aditya Sivakumar Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: [illegible]

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \rightarrow (-1,0) \rightarrow (-1,1) \rightarrow (0,2) \rightarrow (1,1) \rightarrow (1,0) \quad .$$

Explain!

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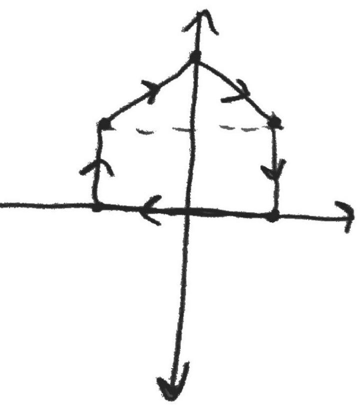
ans. -18



$$\int_C F \cdot dr = \iint_D 11 - 5 \, dA$$

$$= \iint_D 6 \, dA = (\text{Integrand})(\text{Area of } D)$$

$$\text{Area}_D = (1)(2) + \left(\tfrac{1}{2}\right)(2)(1) = 3$$

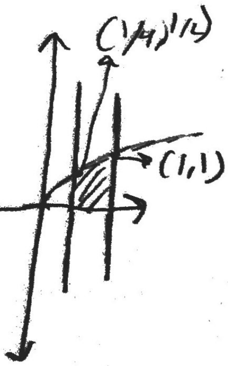
$$(\text{Integrand})(\text{Area}_D) = (6)(3) = 18$$

Because clockwise: $(-1)(18) = \boxed{-18}$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,\cancel{dxdy}\ dydx \ .$$

ans. $\int_0^1\int_0^{y^2+1/2} f(x,y)\,dxdy$



$y^2 \le x \le y^2+\frac{1}{2}$

$0 \le y \le 1$

$$\int_{1/4}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,dydx$$

$$\boxed{=\int_0^1\int_0^{y^2+\frac{1}{2}} f(x,y)\,dx\,dy}$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

ans. $z = \frac{7\pi}{18} - \frac{5}{6}y - \frac{1}{2}x$

$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) - 7 = 0$

$\frac{\partial F}{\partial x}(a,b,c)(x-a) + \frac{\partial F}{\partial y}(a,b,c)(y-b) + \frac{\partial F}{\partial y}(a,b,c)(z-c) = 0$

$[-2\sin(x+y) - 4\sin(x+z)](x-\pi/6) +$

$[-2\sin(x+y) - 8\sin(y+z)](y-\pi/6) +$

$[-4\sin(x+z) - 8\sin(y+z)](z-\pi/6) = 0$

$= (-6\sin\pi/3)(x-\pi/6) +$

$(-10\sin(\pi/3))(y-\pi/6) +$

$(-12\sin\pi/3)(z-\pi/6) = 0$

$= -3\sqrt{3}(x-\pi/6) - 5\sqrt{3}(y-\pi/6) - 6\sqrt{3}(z-\pi/6) = 0$

$-3x + 3\pi/6 - 5y + 5\pi/6 - 6z + 6\pi/6 = 0$

$-3x - 5y - 6z = -14\pi/6$

$3x + 5y + 6z = 14\pi/6$

$\boxed{z = \frac{14\pi}{36} - \frac{5}{6}y - \frac{1}{2}x}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

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ans. $\langle 0, 0, 0 \rangle$.

$a \times (b + c) = a \times b + a \times c$

$a \times (b + c) = \langle 3, 2, 1 \rangle$

I don't know how to do this

Dr. Z sorry

5. **(12 points) Find the three angles of the triangle ABC where**

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

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ans. The angle at A is: $\pi/3$ radians ;

The angle at B is: $\pi/3$ radians ;

The angle at C is: $\pi/3$ radians ;

$\vec{AB} = \langle 1,0,1 \rangle$ $\qquad \cos\theta = \dfrac{A \cdot B}{|A||B|}$

$\vec{AC} = \langle 1,1,0 \rangle$

$\cos\theta = \dfrac{1}{(\sqrt{2})^2} = \dfrac{1}{2} \quad \cos^{-1}(1/2) = \theta_A = \dfrac{\pi}{3}$

$\vec{BA} = \langle -1,0,1 \rangle$ $\qquad \cos\theta = \dfrac{A \cdot B}{|A||B|}$

$\vec{BC} = \langle 0,1,-1 \rangle$

$\cos\theta = \dfrac{1}{(\sqrt{2})^2} \qquad \cos^{-1}(1/2) = \theta_B = \dfrac{\pi}{3}$

$\theta_C = \pi - \theta_A - \theta_B = \pi - \dfrac{\pi}{3} - \dfrac{\pi}{3} = \dfrac{\pi}{3} = \theta_C$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

ans. $\langle -4/\sqrt{3}, -4/\sqrt{3}, -4/\sqrt{3} \rangle$

$\nabla f = \langle 3x^2+yz, 3y^2+xz, 3z^2+xy \rangle = \langle 4,4,4 \rangle$

$v = \langle -1,-1,-1 \rangle - \langle 1,1,1 \rangle$

$= \langle -2,-2,-2 \rangle$ $\quad u = \dfrac{\langle -2,-2,-2 \rangle}{\sqrt{12}}$ $\quad = \langle -1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3} \rangle$

$D_u f = \nabla f \cdot u = \langle 4,4,4 \rangle \cdot \langle -1/\sqrt{3}, -1/\sqrt{3}, -1/\sqrt{3} \rangle$

$$\boxed{= \langle 4/\sqrt{3}, -4/\sqrt{3}, -4/\sqrt{3} \rangle}$$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

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ans. $6 \cos 2$

$$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial u}$$

$$= (6x)\, e^u \cos v + (-6y)(e^u \sin v)$$

$$= 6(e^u \cos v)^2 - 6(e^u \sin v)^2$$

$$= 6 \cos^2 1 - 6 \sin^2 1$$

$$= 6\,(\cos^2(1) - \sin^2(1))$$

$$\boxed{= 6 \cos(2)}$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + \cos(y^3 + yz)\,,\ -2y + e^{x+z^2}\,,\ 5z + \sin(xy^3 + e^x)\,\rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

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ans. 8π

$$\int_S F \cdot ds = \iiint_V div(F)\, dv = \iiint_V 6\, dv$$

$$div(F) = 3 + (-2) + 5 = 6$$

$$= (6)(\text{Volume}_V)$$

$$\text{Volume}_V = \left(\frac{4}{3}\right)(\pi)(r^3)\left(\frac{1}{8}\right) = \frac{4\pi}{3}(2^3)\left(\frac{1}{8}\right) = \frac{4\pi}{3}$$

$$(6)\left(\frac{4\pi}{3}\right) = \boxed{8\pi}$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z\,,\, 2x\,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

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ans. -15

$$\iint_S F.dS = \iint_D -3z(2) - 2x(3) + y + z \, dA$$

$$= \int_0^1\int_0^1 -6(2x+3y) - 6x + y + 2x + 3y \, dA$$

$$= \int_0^1\int_0^1 -12x - 18y - 6x + y + 2x + 3y \, dy\, dx$$

$$= \int_0^1\int_0^1 -16x - 14y \, dy\, dx$$

$$= -\int_0^1\int_0^1 16x + 14y \, dy\, dx$$

$$= -\int_0^1 16xy + 7y^2\Big|_0^1 dx$$

$$= -\int_0^1 16x + 7 \, dx$$

$$= -\left(8x^2 + 7x\Big|_0^1\right)$$

$$\boxed{= -15}$$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

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ans. $(3/4, -1)$; Saddle Point

$f_x = 4 - \dfrac{2}{2x+y}$ $\quad 4 - \dfrac{2}{2x+y} = 0 \quad 4 = \dfrac{2}{2x+y}$

$\frac{1}{2} = 2x+y$ $\quad \frac{1}{2} = -\frac{1}{2y}$ $\quad y = -1$

$f_y = -2y - \dfrac{1}{2x+y}$ $\quad -2y - \dfrac{1}{2x+y} = 0 \quad -2y = \dfrac{1}{2x+y}$

$-\dfrac{1}{2y} = 2x+y$

$f_{xx} = \dfrac{4}{(2x+y)^2}$

$\frac{1}{2} = 2x - 1$

$f_{yy} = \dfrac{1}{(2x+y)^2} - 2$ $\quad \frac{3}{2} = 2x \quad x = \frac{3}{4}$

$f_{xy} = \dfrac{2}{(2x+y)^2}$ $\quad$ Critical Point: $(3/4, -1)$

$f_{xx}\big|_{(3/4,-1)} = \dfrac{4}{(1/2)^2} = \dfrac{4}{1/4} = 16 > 0$

$D = (16)(2) - (8)^2$

$= 32 - 64$

$= -32 < 0 \rightarrow$ Saddle point

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} .$$

ans. $\frac{9001}{3000}$

$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

$a=1 \quad b=1 \quad c=2$

$f(a,b,c) = 3$

$f_x(a,b,c) = \frac{2x}{\sqrt{2x^2+3y^2+z^2}} = \frac{2}{3}$

$f_y(a,b,c) = \frac{3y}{\sqrt{2x^2+3y^2+z^2}} = \frac{3}{3} = 1$

$f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}} = \frac{2}{3}$

$L(x,y,z) = 3 + \left(\frac{2}{3}\right)\left(\frac{1}{1000}\right) - \frac{1}{1000} + \frac{2}{3}\left(\frac{1}{1000}\right)$

$= 3 + \frac{2}{3000} - \frac{3}{3000} + \frac{2}{3000}$

$= 3 + \frac{1}{3000} = \frac{9001}{3000}$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x\, dy\, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x\, dy\, dx \quad .$$

Explain!

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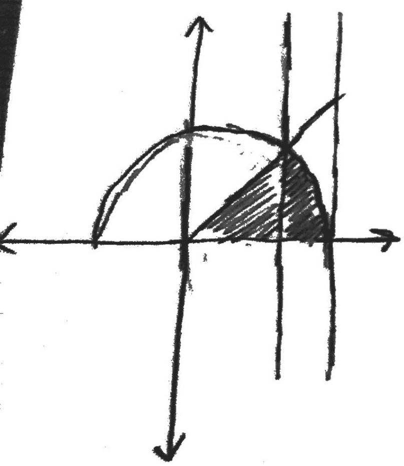
ans. $\frac{\sqrt{2}}{6}$.



$0 \leq r \leq 1$

$0 \leq \theta \leq \frac{\pi}{4}$

$$\int_0^{\pi/4}\int_0^1 r^2 \cos\theta \, dr\, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{3} \cos\theta \, d\theta$$

$$= \frac{1}{3}\left(\sin\theta \Big|_0^{\pi/4}\right)$$

$$= \frac{1}{3}\left(\sqrt{2}/2\right) = \boxed{\frac{\sqrt{2}}{6}}$$

13. (12 points) Convert the triple iterated integral

$$\int_0^4 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to spherical coordinates. Do not evaluate.

ans. $\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^{2} \rho^6 \sin^4\varphi \cos\varphi \sin\theta \cos^2\theta \, d\rho\, d\theta\, d\varphi$

$0 \le \rho \le 2$

$\frac{\pi}{2} \le \theta \le \pi$

$0 \le \varphi \le \frac{\pi}{2}$

$\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^{2} (\rho^2 \sin^2\varphi \cos^2\theta)(\rho \sin\varphi \sin\theta)(\rho\cos\varphi)(\rho^2 \sin\varphi)\, d\rho\, d$

$= \int_0^{\pi/2} \int_{\pi/2}^{2\pi} \int_0^{2} \rho^6 \sin^4\varphi \cos\varphi \sin\theta \cos^2\theta \, d\rho\, d\theta\, d\varphi$

14. (12 points) Find the curvature of the curve

$$r(t) = \langle 5, 3\sin t, 3\cos t \rangle$$

at the point where $t = \frac{\pi}{3}$.

ans: $1/3$

$$k = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

$$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle = \langle 0, 3/2, -3\sqrt{3}/2 \rangle$$

$$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle = \langle 0, -\frac{3\sqrt{3}}{2}, -3/2 \rangle$$

$$|r'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t} = \sqrt{9} = 3$$

$$r'(t) \times r''(t) = \langle -9, 0, 0 \rangle$$

$$|r'(t) \times r''(t)| = \sqrt{81} = 9$$

$$k = \frac{9}{3^3} = \frac{9}{27} \boxed{= \frac{1}{3}}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2\rangle \quad , \quad 0 < u < v < 1 \ .$$

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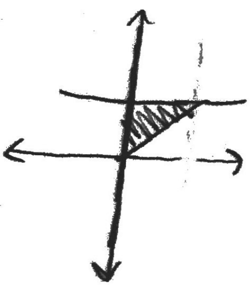
ans. $\int_0^1 \int_u^1 2\sqrt{v^4+4u^2v^2+u^4}\,dv\,du$



$\iint_S dS \qquad dS = |r_u \times r_v|\,du\,dv$

$r_u = \langle 2u, v, 0\rangle \qquad r_u \times r_v = \langle 2v^2, -4uv, 2u^2\rangle$

$r_v = \langle 0, u, 2v\rangle$

$|r_u \times r_v| = \sqrt{4v^4+16u^2v^2+4u^4}$

$$= \int_0^1\int_0^v 2\sqrt{v^4+4u^2v^2+u^4}\,du\,dv$$

$$\boxed{= \int_0^1\int_u^1 2\sqrt{v^4+4u^2v^2+u^4}\,dv\,du}$$

16. (12 points) Let

$$f(x, y, z) = xy^2z^3 \quad ,$$

and let

$$g(x, y, z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point (1, 1, 1).

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ans. 14

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle = \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle = \langle 1, 2, 3 \rangle$

$\langle 1, 2, 3 \rangle \cdot \langle 1, 2, 3 \rangle = 1 + 4 + 9 = \boxed{14}$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w}.$$

ans. DNE

From x-axis:

$$\lim_{\substack{y,z,w=0 \\ (x,y,z,w)\to(0,0,0,0)}} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{x^2}{x} = x$$

From y-axis

$$\lim_{\substack{x,z,w=0 \\ (x,y,z,w)\to(0,0,0,0)}} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} = \frac{y^2}{y} = y$$

No agreements, so limit DNE!