NAME: (print!) ABHIJIT KULKARNI RUID: (print!) 193000621

SSC: (circle) None / I / II / I and II 4 different final grade calculations!

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

---

1. 1
2. $\int_{1/4}^{1}\int_{\sqrt{y}}^{1} f(x,y)\,dy\,dx$
3. $x+y+1$
4. 12
5. $1\,;\,1\,;\,1$
6. $-12\sqrt{3}$
7. $(\cos 1)6x - (\sin 1)6y$
8. 1
9. -15
10. $(0,1)$ local min ; $(1,0)$ local max
11. $9001/3000$
12. $1/3\sqrt{2}$
13. $\int_0^{\pi}\int_0^{\pi}\int_1^{2} \rho^4 \sin^3\phi \cos\phi \sin\theta \cos\theta \; d\rho\, d\theta\, d\phi$
14. 3
15. $\int_0^1\int_0^1 u^3 v^3\, du\, dv$
16. 14
17. DNE

Sign the following declaration:

I Abhijit Kulkarni Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: AKulkarni

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (cos\,(e^{\sin x}) + 5y)\,dx \;+\; (sin\,(e^{\cos y}) \;+\; 11x\,)\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

---

ans. 1

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$$

1 x 1 1

0 0 0 y

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}\,=x}^{1\,=x} \int_{0\,=y}^{\sqrt{x}\,=y} f(x,y)\, \cancel{dx\,dy}\ dy\,dx\ .$$

---

ans. $\int_{1/4}^{1} \int_{\sqrt{y}}^{1} f(x,y)\, dx\, dy$

$1/4 \le x \le 1$ $\qquad \le y \le$

$\Rightarrow$

$0 \le y \le \sqrt{x}$ $\qquad \le x \le$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \quad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z =$ X + y + 1

---

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

$(1, 1, -1)$ $(1, -1, 1)$ $(2, 1, 2)$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

3 × 4

ans. 12

$$b \times c = \begin{vmatrix} i & j & k \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = i(b_y c_z - b_z c_y) - j(b_x c_z - b_z c_x) + k(b_x c_y - b_y c_x)$$

1 1 1

$$a \times b = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = i(a_y b_z - a_z b_y) - j(a_x b_z - a_z b_x) + k(a_x b_y - a_y b_x)$$

1 -1 -1

$$a \times c = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix} = i(a_y c_z - a_z c_y) - j(a_x c_z - a_z c_x) + k(a_x c_y - a_y c_x)$$

2 -1 2

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: | radians ;

The angle at B is:) radians ;

The angle at C is:) radians ;

$d = \sqrt{(\quad}$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

---

ans. $-12\sqrt{3}$

$\nabla f = \langle 3x^2 + yz \, , \; xz + 3y^2 \, , \; xy + 3z^2 \rangle$

$|\langle -1, -1, -1 \rangle| = \sqrt{3}$

$u = \sqrt{3} \langle -1, -1, -1 \rangle = \langle -\sqrt{3}, -\sqrt{3}, -\sqrt{3} \rangle$

$\nabla f (1, 1, 1) = \langle 4, 4, 4 \rangle$

$\nabla f \cdot u = \langle 4, 4, 4 \rangle \cdot \langle -\sqrt{3}, -\sqrt{3}, -\sqrt{3} \rangle$

$= -12\sqrt{3}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\cancel{\frac{\partial g}{\partial u}} \quad \frac{dz}{ds}$$

at $(\cancel{u}, \cancel{v}) = (0, 1)$ (s,t), where

$$z\,\cancel{g}(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^{\cancel{u}\,s} \cos \cancel{v}\,t \ , \quad y = e^{\cancel{u}\,s} \sin \cancel{v}\,t \, .$$

---

ans. $(\cos 1)\, 6x \; - \; (\sin 1)\, 6y$

$\frac{dz}{dx} = 6x \quad , \quad \frac{dz}{dy} = -6y$

$\frac{dx}{ds} = e^s \cos t \quad , \quad \frac{dy}{ds} = e^s \sin t$

$\frac{dx}{dt} = -e^s \sin t \quad , \quad \frac{dy}{dt} = e^s \cos t$

$\frac{dz}{ds} = (6x)(e^s \cos t) + (-6y)(e^s \sin t)$

$\frac{dz}{ds} = 6xe^s \cos t - 6ye^s \sin t$

$\frac{dz}{ds}(s=0, t=1) = 6xe^0 \cos 1 - 6ye^0 \sin 1$

$= 6x \cos 1 - 6y \sin 1$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle \underbrace{3x + cos(y^3 + yz)}_{P} \, , \, \underbrace{-2y + e^{x+z^2}}_{Q} \, , \, \underbrace{5z + \sin(xy^3 + e^x)}_{R} \rangle \quad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. 1

Divergence Thm.

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z \,,\, 2x \,,\, y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

ans. -15

$$\iint_S F \cdot dS = \iint_D \left(-P \frac{dg}{dx} - Q \frac{dg}{dy} + R\right) dA$$

$P = 3z \quad , \quad Q = 2x \quad , \quad R = y+z \quad ; \quad g = 2x+3y$

$$= \iint_D (-3z(2) - 2x(3) + y + z)\, dA = \iint_D (-5z - 6x + y)\, dA$$

$$= \iint_D (-5(2x+3y) - 6x + y)\, dA = \iint_D -16x - 14y \; dA = \int_0^1 \int_0^1 -16x - 14y \; dx\,dy$$

$-10x - 15y - 6x + y$

$-16x - 14y$

$= -15$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x,y) = 4x - y^2 - \ln(2x+y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

---

ans. (0, 1) local min ; (1, 0) local max

$$f_x = -\frac{2}{2x+y} + 4$$

$$f_y = -2y - \frac{1}{2x+y} \quad \Rightarrow \quad 0 = -2y - \frac{1}{2x+y}$$

$$0 = -2y(2x+y) - 1$$

$$0 = -4xy - 2y^2 - 1$$

?

$$f_{xx} = \frac{4}{(2x+y)^2}$$

$$f_{xy} = \frac{2}{(2x+y)^2}$$

$$f_{yy} = -2 + \frac{1}{(2x+y)^2}$$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1,1,2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x,y,z) = \sqrt{2x^2+3y^2+z^2} \quad .$$

---

ans. 9001/3000

$$f_x = \frac{2x}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_y = \frac{3y}{\sqrt{2x^2+3y^2+z^2}}$$

$$f_z = \frac{z}{\sqrt{2x^2+3y^2+z^2}}$$

Cont. at $(1,1,2)$

$f(1,1,2) = 3$

$f_x(1,1,2) = 2/3$

$f_y(1,1,2) = 3/3 = 1$

$f_z(1,1,2) = 2/3$

$$L(1,1,2) = f(1,1,2) + f_x(1,1,2)(x-1) + f_y(1,1,2)(y-1) + f_z(1,1,2)(z-2)$$

$$= 3 + \frac{2}{3}(x-1) + (y-1) + \frac{2}{3}(z-2)$$

$$f(1.001, 0.999, 2.001) = 3 + \frac{2}{3}(0.001) - 0.001 + \frac{2}{3}(0.001)$$

$$= \frac{9001}{3000}$$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^{x} x \, dy \, dx + \int_{\frac{\sqrt{2}}{2}}^{1} \int_0^{\sqrt{1-x^2}} x \, dy \, dx \quad .$$

Explain! $\frac{1}{6\sqrt{2}} + \frac{1}{6\sqrt{2}} = \frac{2}{6\sqrt{2}} = \frac{1}{3\sqrt{2}}$

---

ans. $\frac{1}{3\sqrt{2}}$

---

13. (12 points) Convert the triple iterated integral

$$\int_0^{2}\int_0^{\sqrt{4-z^2}}\int_{-\sqrt{4-z^2-y^2}}^{0} x^2\,y\,z\,dx\,dy\,dz$$

to **spherical coordinates. Do not evaluate.**

---

ans. $\int_0^{\pi}\int_0^{\pi}\int_1^{2} \rho^4 \sin^3\phi \cos\phi \sin\theta \cos\theta \; d\rho\, d\theta\, d\phi$

---

radii $= 2$

$x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$,

$z = \rho\cos\phi$;

$(\rho\sin\phi\cos\theta)(\rho\sin\phi\cos\theta) = \rho^2\sin^2\phi\cos\theta$

$(\rho^2\sin^2\phi\cos\theta)(\rho\sin\phi\sin\theta) = \rho^3\sin^3\phi\sin\theta\cos\theta$

$(\rho^3\sin^3\phi\sin\theta\cos\theta)(\rho\cos\phi) = \rho^4\sin^3\phi\cos\phi\sin\theta\cos\theta$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

---

ans. 3

$r'(t) = \langle 0, 3\cos t, -3\sin t \rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t \rangle$

$$r'(t) \times r''(t) = \begin{vmatrix} i & j & k \\ 0 & 3\cos t & -3\sin t \\ 0 & -3\sin t & -3\cos t \end{vmatrix} = i \begin{vmatrix} 3\cos t & -3\sin t \\ -3\sin t & -3\cos t \end{vmatrix}$$

$$= (-9\cos^2 t - 9\sin^2 t)\, i = \left(-9(\cos^2 t + \sin^2 t)\right) i = -9i$$

$$= \langle -9, 0, 0 \rangle$$

$|r'(t) \times r''(t)| = \sqrt{(-9)^2 + 0^2 + 0^2} = \sqrt{81} = 9$

$|r'(t)| = \sqrt{0^2 + (3\cos t)^2 + (-3\sin t)^2} = \sqrt{9\cos^2 t + 9\sin^2 t} = \sqrt{9} = 3$

$$\kappa(t) = \frac{|r'(t) \times r''(t)|}{|r'(t)|} = \frac{9}{3} = 3$$

$\kappa\left(\frac{\pi}{3}\right) = 3$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2 , uv , v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_0^1 u^3 v^3 \, du \, dv$

---

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x + y^2 + z^3 \quad .$$

compute the dot-product

$$grad(f) \, . \, grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14

$\nabla f = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle \Rightarrow \nabla f(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla g = \langle 1, 2y, 3z^2 \rangle \Rightarrow \nabla g(1,1,1) = \langle 1, 2, 3 \rangle$

$\nabla f_{(1,1,1)} \cdot \nabla g_{(1,1,1)} = \langle 1,2,3 \rangle \cdot \langle 1,2,3 \rangle = 1 + 4 + 9 = 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad . \quad = \text{"}\frac{0}{0}\text{"}$$

---

ans. DNE

---

4 variables in a limit problem is nonsense!