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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as `finalFirstLast.pdf` to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 18
2. $\int_0^1 \int_{y^2}^1 f(x,y)\, dx\, dy$
3. $z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{8\sqrt{3}\pi + \pi}{6}$
4. $3i - 6j + 9k$
5. $\pi/3, \pi/3, \pi/3$
6. $-24/\sqrt{12}$
7. $6\cos^2 1 - 6\sin^2 1$
8. $4/3\, \pi$
9. -15
10. $(\frac{3}{4}, -1)$ is a saddle pt.
11. 3.0003
12. $\sqrt{2}/6$
13. $\int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^2 \rho^6 \sin^4\phi \cos\phi \cos^2\theta \sin\theta \, d\rho\, d\phi\, d\theta$
14. $1/3$
15. $\int_0^1\int_u^1 2\sqrt{v^4 + 4u^2v^2 + u^4}\, dv\, du$
16. 14
17. exists at 0

Sign the following declaration:

I Aayushi Kasera Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Aayushi Kasera

Possibly useful facts from school Geometry (that you are welcome to use) : (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x, y)$, above the region of the xy-plane , D, then

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C \overbrace{(cos\,(e^{\sin x}) + 5y)}^{P}\,dx \;+\; \overbrace{(sin\,(e^{\cos y}) \;+\; 11x\,)}^{Q}\,dy \quad ,$$

over the path consisiting of the five line segments (in that order)

$$(1,0) \to (-1,0) \to (-1,1) \to (0,2) \to (1,1) \to (1,0) \quad .$$

Explain!

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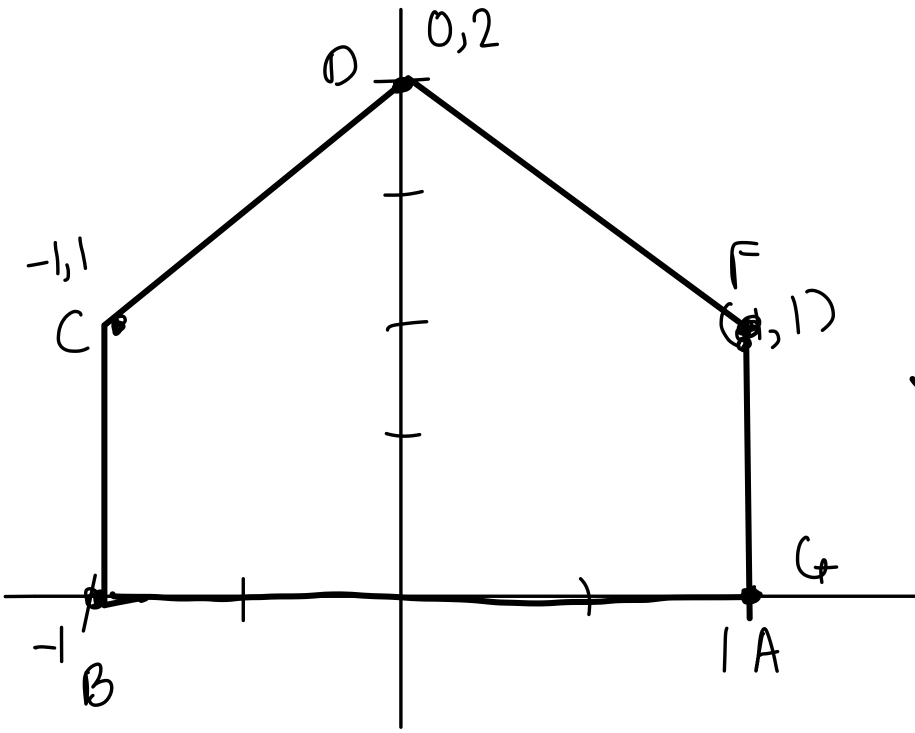
ans. 18



$$\iint \left[\frac{dQ}{dx} - \frac{dP}{dy}\right]$$

$$\iint (11 - 5)\,dA$$

$$6 \iint dA$$

$$6 \times 3$$

$$= 18$$

$$A = 2\,\frac{(b_1+b_2)\,h}{2}$$

$$\cancel{2}\,\frac{(2+1)\,1}{\cancel{2}}$$

$$= 3$$

2. (12 points) Change the order of integration

$$\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\, d\cancel{x\,dy}\quad dy\,dx$$

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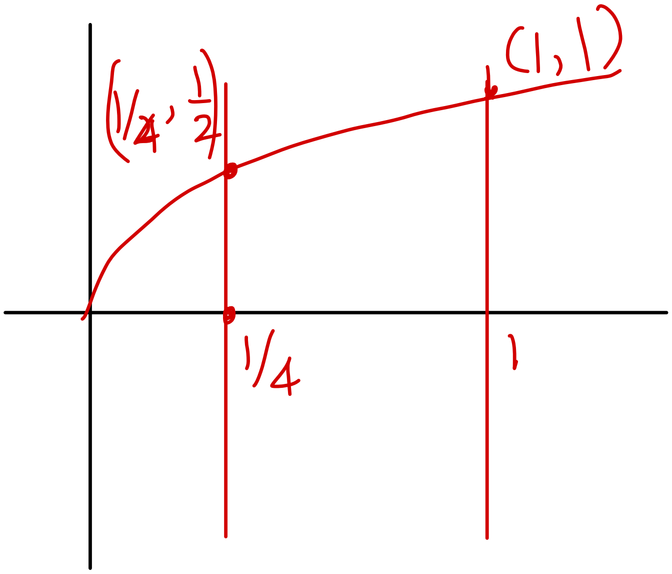
ans. $$\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$$



$y = \sqrt{x}$

$y = 0$

$x = 1$

$x = \frac{1}{4}$

$x = y^2$

$$\int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$$

3. (12 points) Find the equation of the tangent plane at the point $(\frac{\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6})$ to the surface given implicitly by

$$2\cos(x+y) + 4\cos(x+z) + 8\cos(y+z) = 7 \qquad .$$

Express you answer in **explicit** form, i.e in the format $z = ax + by + c$.

---

ans. $z =$ $-3\sqrt{3}x - 5\sqrt{3}y + \frac{8\sqrt{3}\pi + \pi}{6}$

$f_x = -2\sin(x+y) - 4\sin(x+z)$

$= -2\sin(\frac{\pi}{3}) - 4\sin(\frac{\pi}{3})$

$-\sqrt{3} - 2\sqrt{3}$

$= -3\sqrt{3}$

$f_y = -2\sin(x+y) - 8\sin(y+z)$

$= -\sqrt{3} - 4\sqrt{3}$

$= -5\sqrt{3}$

$z - \frac{\pi}{6} = f_x(x - \frac{\pi}{6}) + f_y(y - \frac{\pi}{6})$

$z = -3\sqrt{3}x + \frac{3\sqrt{3}\pi}{6} - 5\sqrt{3}y + \frac{5\sqrt{3}\pi}{6} + \frac{\pi}{6}$

$z = -3\sqrt{3}x - 5\sqrt{3}y + \frac{8\sqrt{3}\pi + \pi}{6}$

4. (16 points) Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three vectors such that

$$\mathbf{a} \times \mathbf{b} = \mathbf{i} + \mathbf{j} - \mathbf{k} \quad , \quad \mathbf{b} \times \mathbf{c} = \mathbf{i} - \mathbf{j} + \mathbf{k} \quad , \quad \mathbf{a} \times \mathbf{c} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k} \quad .$$

What is

$$(\mathbf{a} + \mathbf{b} + \mathbf{c}) \times (2\mathbf{a} - \mathbf{b} + 3\mathbf{c}) \qquad ?$$

---

ans. $3i - 6j + 9k$

Theorem:-

$$\boxed{u \times (v + w) = u \times v + u \times w}$$

$$au \times v = a(u \times v) = u \times av$$

$$(a + b + c) \times (2a - b + 3c)$$

$$= \cancel{(a \times 2a)} + (a \times -b) + (a \times 3c) +$$
$$\cancel{(b \times b)} + (b \times 2a) + (b \times 3c) +$$
$$(c \times 2a) + (c \times -b) + \cancel{(c \times 3c)}$$

$$= (-\cancel{i} - j + \cancel{k}) + (6i + \cancel{3j} + \underline{6k}) + (-\cancel{2i} - 2j + \cancel{2k})$$
$$+ (\cancel{3}i - \cancel{3j} + \underline{3k}) + (-4i - 2j - \cancel{4k})$$
$$+ (i - \underline{j} + \cancel{k})$$

$$3i - 6j + 9k$$

5. (12 points) Find the three angles of the triangle ABC where

$$A = (0,0,0) \quad , \quad B = (1,0,1) \quad , \quad C = (1,1,0) \quad .$$

---

ans. The angle at A is: radians ; $\pi/3$

The angle at B is: radians ; $\pi/3$

The angle at C is: radians ; $\pi/3$

For Angle A $|\vec{AB}| = \sqrt{2}$ $|\vec{AC}| = \sqrt{2}$

AB AC

$$\frac{\langle 1,0,1\rangle \cdot \langle 1,1,0\rangle}{|\vec{AB}| \cdot |\vec{AC}|}$$

$$\cos^{-1}\left(\frac{1}{2}\right)$$

$= 60^\circ$

For Angle B

BA BC

$\langle -1,0,-1\rangle$ $\langle 0,1,-1\rangle$

$|\vec{BA}| = \sqrt{2}$ $|\vec{BC}| = \sqrt{2}$

$$\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$$

For Angle C

CB CA

$\langle 0,-1,1\rangle$ $\langle -1,-1,0\rangle$

$|CB| = \sqrt{2}$ $|\vec{CA}| = \sqrt{2}$

$$\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$$

$= 60^\circ$

Degrees to radians

$$\cancel{60}\left(\frac{\pi}{\cancel{180}_{3}}\right)$$

6. (12 points) Find the directional derivative of

$$f(x, y, z) = x^3 + y^3 + z^3 + xyz \quad ,$$

at the point $(1, 1, 1)$ in a direction pointing to the point $(-1, -1, -1)$.

ans. $\frac{-24}{\sqrt{12}}$

$v = (-1,-1,-1) - (1,1,1)$

$\langle -2, -2, -2 \rangle$

$|v| = \sqrt{12}$

$u = \frac{-2}{\sqrt{12}} \langle 1, 1, 1 \rangle$

$\nabla f = \langle 3x^2 + yz, 3y^2 + xz, 3z^2 + xy \rangle$

$= \langle 3+1, 3+1, 3+1 \rangle$

$= \langle 4, 4, 4 \rangle$

$= 4 \langle 1, 1, 1 \rangle$

$-\frac{8}{\sqrt{12}} \langle 1, 1, 1 \rangle \cdot \langle 1, 1, 1 \rangle = \boxed{\frac{-24}{\sqrt{12}}}$

7. (12 points) Using the Chain Rule (no credit for other methods), find

$$\frac{\partial g}{\partial u}$$

at $(u, v) = (0, 1)$, where

$$g(x, y) = 3x^2 - 3y^2 \quad ,$$

and

$$x = e^u \cos v \quad , \quad y = e^u \sin v \quad .$$

---

ans. $6\cos^2 1 - 6\sin^2 1$

$$\frac{dg}{du} = \frac{dg}{dx}\frac{dx}{du} + \frac{dg}{dy}\frac{dy}{du}$$

$$6x(e^u \cos v) + (-6y)(e^u \sin v)$$

$$= 6x\cos(1) - 6y\sin(1)$$

$$= 6\cos^2 1 - 6\sin^2 1$$

8. (12 points) Without using Maple (or any other software), compute the vector-field surface integral $\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle\, 3x + cos(y^3 + yz)\,,\, -2y + e^{x+z^2}\,,\, 5z + \sin(xy^3 + e^x)\,\rangle \qquad ,$$

and S is the closed surface in 3D space bounding the region

$$\{(x, y, z) : x^2 + y^2 + z^2 < 4 \quad and \quad x > 0 \quad and \quad y < 0 \quad and \quad z > 0\} \quad .$$

---

ans. $4\pi/3$

Using divergence

$\operatorname{div}(F) = P_x + Q_y + R_z$

$= 3 - 2 + 5$

$= 6$

$\because z > 0$

$\phi = 0 .. \frac{\pi}{2}$

$\theta = \frac{\pi}{2} .. \pi$

$x^2 + y^2 + z^2 < 4$

$\rho^2 < 4$

$0 < \rho < 2$

$$\int_{\pi/2}^{\pi} \int_0^{\pi/2} \int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\int_{\pi/2}^{\pi} \int_0^{\pi/2} \left[\frac{\rho^3}{3} \sin\phi\right]_0^2$$

$$= \int_{\pi/2}^{\pi} \left[-\frac{8}{3} \cos\phi\right]_0^{\pi/2}$$

$$= \left[\frac{8}{3}\theta\right]_{\pi/2}^{\pi} =$$

$$= \frac{8}{3}\pi - \frac{4}{3}\pi$$

$$= \frac{4}{3}\pi$$

9. (12 points) Compute the vector-field surface integral $\int\int_S \mathbf{F}.d\mathbf{S}$ if

$$\mathbf{F} = \langle 3z , 2x , y+z \rangle \quad ,$$

and S is the oriented surface

$$z = 2x + 3y \quad , \quad 0 < x < 1, \quad 0 < y < 1 \quad ,$$

with **upward pointing** normal.

---

ans. -15

$P = 3z$

$Q = 2x$

$R = y + z$

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \quad .$$

$g(x,y) = 2x + 3y$

$\frac{dg}{dx} = 2 \qquad \frac{dg}{dy} = 3$

$\int\int_D -6z - 6x + (y+z)\, dA$

$\int\int -5z - 6x + y$

$\int\int -5(2x+3y) - 6x + y$

$\int\int -10x - 15y - 6x + y$

$\int\int -16x - 14y$

$\int\int -16x - 14y$

$\int_0^1\int_0^1 -16x - 14y \; dy\, dx$

int(int(-16x-14y

y=0..1),

x=0..1);

$= -15$

10. (12 points) Without using Maple or software, find the critical point(s) of

$$f(x, y) = 4x - y^2 - \ln(2x + y) \quad ,$$

and decide for each whether it is a local maximum, local minimum, or saddle point. Explain.

ans. $\left(\frac{3}{4}, -1\right)$ is a saddle point

$$f_x = 4 - \frac{2}{2x+y} \qquad f_y = -2y - \frac{1}{2x+y}$$

$$2 = \frac{1}{2x+y} \qquad \frac{1}{2x+y} = -2y$$

$$2x+y = \frac{1}{2} \qquad 1 = -2y(2x+y)$$

$$1 = -4xy - 2y^2$$

$$4x + 2y = 1 \qquad 2y^2 + 4xy + 1 = 0$$

$$x = \frac{1-2y}{4}$$

$$f_{xx} = \frac{4}{(2x+y)^2} = 16$$

$$x = \frac{3}{4}, \; y = -1$$

$$f_{yy} = -2 + \frac{1}{(2x+y)^2} = 2$$

$$f_{xy} = \frac{2}{(2x+y)^2} = 8$$

$$D = 32 - 64 < 0$$

$\therefore$

11. (12 points) Without using Maple or software, using a **Linearization** around the point $(1, 1, 2)$, approximate $f(1.001, 0.999, 2.001)$ if

$$f(x, y, z) = \sqrt{2x^2 + 3y^2 + z^2} \quad .$$

---

ans. 3.0003

$f(1,1,2) = \sqrt{2+3+4}$

$= 3$

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= \left\langle \frac{4x}{2\sqrt{2x^2+3y^2+z^2}}, \frac{6y}{2\sqrt{2x^2+3y^2+z^2}}, \frac{2z}{2\sqrt{2x^2+3y^2+z^2}} \right\rangle$

$= \left\langle \frac{2}{3}, 1, \frac{2}{3} \right\rangle$

$= \left\langle \frac{4}{3}, 1, \frac{2}{3} \right\rangle$

$L(x,y) = 3 + \frac{2}{3}(+0.001) + 1(-0.001) + \frac{2}{3}(0.001)$

$3 + \left(\frac{4}{3} - 1\right)0.001$

$= 3.0003$

12. (12 points) Without using Maple (or any other software) and by using polar coordinates (no credit for doing it directly) find

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx + \int_{\frac{\sqrt{2}}{2}}^1 \int_0^{\sqrt{1-x^2}} x\,dy\,dx \quad . \quad \frac{\sqrt{2}}{12}$$

Explain!

---

ans. $\frac{\sqrt{2}}{6}$

$$\int_0^{\frac{\sqrt{2}}{2}} \int_0^x x\,dy\,dx$$

$$\int_0^{\sqrt{2}/2} [xy]_0^x \, dx$$

$$\int_0^{\sqrt{2}/2} x^2 \, dx$$

$$\left[\frac{x^3}{3}\right]_0^{\sqrt{2}/2}$$

$$\frac{\cancel{2}^{1}\sqrt{2}}{\cancel{8}_{4} \times 3} = \frac{\sqrt{2}}{12}$$

$y = \sqrt{1-x^2}$

$x^2 + y^2 = 1$

$r = 0 .. 1$

$\theta = 0 .. \pi$

$$\int_0^{\pi/2} \int_0^{\sqrt{2}/2} r^2 \cos\theta \, dr\, d\theta$$

$$\int_0^{\pi/2} \left[\frac{r^3}{3} \cos\theta\right]_0^{\frac{\sqrt{2}}{2}}$$

$$\int_0^{\pi/2} \frac{\sqrt{2}}{12} \cos\theta$$

$$\left[\frac{\sqrt{2}}{12} \sin\theta\right]_0^{\pi/2}$$

$$= \frac{\sqrt{2}}{12}$$

$$\therefore \frac{\sqrt{2}}{12} + \frac{\sqrt{2}}{12}$$

$$= \frac{2\sqrt{2}}{12} = \frac{\sqrt{2}}{6}$$

Check

13. (12 points) Convert the triple iterated integral

$$\int_0^{2}\int_0^{\sqrt{4-z^2}}\int_{-\sqrt{4-z^2-y^2}}^{0} x^2\, y\, z\, dx\, dy\, dz$$

to **spherical coordinates**. **Do not evaluate.**

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ans. $\int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^6 \sin^4\phi \cos\phi \cos^2\theta \sin\theta \, d\rho\, d\phi\, d\theta$

$-\sqrt{4-z^2-y^2} \le x \le 0 \qquad 0 \le y \le \sqrt{4-z^2}$

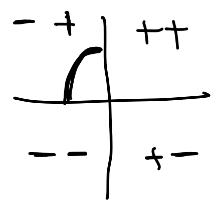
$0 \le z \le 4$

z is positive, $z \ge 0$

$\therefore \phi :- \quad 0 \le \phi \le \pi/2$

$y \ge 0$ and $x \le 0$

$\theta = \pi/2 \, .. \, \pi$



$0 \le \rho \le 2$

$$\int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^2 \sin^2\phi \cos^2\theta\, \rho \sin\phi \sin\theta\, \rho\cos\phi\, \rho^2 \sin\phi$$

$$\int_{\pi/2}^{\pi}\int_0^{\pi/2}\int_0^{2} \rho^6 \sin^4\phi \cos\phi \cos^2\theta \sin\theta\, d\rho\, d\phi\, d\theta$$

14. (12 points) Find the curvature of the curve

$$\mathbf{r}(t) = \langle 5, 3\sin t, 3\cos t\rangle$$

at the point where $t = \frac{\pi}{3}$.

ans. $1/3$

$$r(t) = \langle 5, 3\sin(t), 3\cos(t)\rangle$$

$$r'(t) = \langle 0, 3\cos t, -3\sin(t)\rangle$$

$$= \left\langle 0, \frac{3}{2}, -\frac{3\sqrt{3}}{2}\right\rangle$$

$$r''(t) = \langle 0, -3\sin(t), -3\cos(t)\rangle$$

$$= \left\langle 0, -\frac{3\sqrt{3}}{2}, -\frac{3}{2}\right\rangle$$

$$r'(t) \times r''(t) = -9i + 0j + 0k$$

$$\begin{vmatrix} i & j & k \\ 0 & \frac{3}{2} & -\frac{3\sqrt{3}}{2} \\ 0 & -\frac{3\sqrt{3}}{2} & -\frac{3}{2} \end{vmatrix}$$

$$|r'(t) \times r''(t)| = 9$$

$$|r'(t)| = \sqrt{\frac{9}{4} + \frac{27}{4}}$$

$$= 3$$

$$|r'(t)|^3 = 27$$

$$K = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} = \frac{\cancel{9}^{1}}{\cancel{27}_{3}} = \frac{1}{3}$$

15. (12 points) Set-up an iterated double integral, in type I format, but do not compute, for the surface area of the surface given parameterically by

$$\mathbf{r}(u,v) = \langle u^2\,,\, uv\,,\, v^2 \rangle \quad , \quad 0 < u < v < 1 \quad .$$

---

ans. $\int_0^1 \int_u^1 2\sqrt{v^4 + 4u^2v^2 + u^4}\, dv\, du$

$r_u = \langle 2u, v, 0 \rangle$

$r_v = \langle 0, u, 2v \rangle$

$(u^2 + v^2)^2$

$u^4 + v^4 + 2u^2v^2$

$|r_u \times r_v| = 2v^2 i - 4uv j + 2u^2 k$

$= \sqrt{4v^4 + 16u^2v^2 + 4u^4}$

$= 2\sqrt{v^4 + 4u^2v^2 + u^4}$

$$\int_0^1 \int_u^1 2\sqrt{v^4 + 4u^2v^2 + u^4}\, dv\, du$$

1

16. (12 points) Let

$$f(x,y,z) = xy^2z^3 \quad ,$$

and let

$$g(x,y,z) = x+y^2+z^3 \quad .$$

compute the dot-product

$$grad(f)\,.\,grad(g) \quad .$$

at the point $(1,1,1)$.

---

ans. 14

$f = xy^2z^3$

$\nabla f = \langle f_x, f_y, f_z \rangle$

$= \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$

$= \langle 1, 2, 3 \rangle$

$\nabla g = \langle g_x, g_y, g_z \rangle$

$= \langle 1, 2y, 3z^2 \rangle$

$= \langle 1, 2, 3 \rangle$

Ans $= \langle 1,2,3 \rangle . \langle 1,2,3 \rangle$

$= 1+4+9$

$= 14$

17. (8 points) Decide whether the following limit exists. If it does ,find it. If it does not, explain why it does not exist.

$$\lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2-(z+w)^2}{x+y-z-w} \quad .$$

ans. exists at 0

$y = ax$

$z = bx$

$w = cx$

$(x+y)^2 - (z+w)^2$

$= (x+y+z+w)(x+y-z-w)$

$$\lim_{(x,y,z,w)\to 0,0,0,0} \frac{(x+y+z+w)\cancel{(x+y-z-w)}}{\cancel{(x+y-z-w)}}$$

$= 0$