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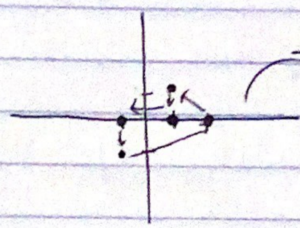
Multivariable calculus Final Exam

1.) $\int_C (\cos(e^{\sin x}) + 5y)\,dx + (\sin(e^{\cos y}) + 11x)\,dy$

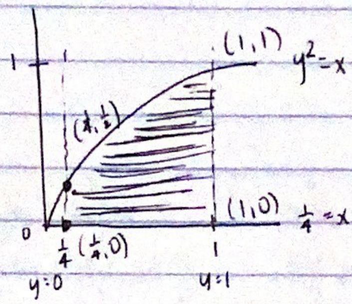
$P_y = 5$ $Q_x - P_y = 11 - 5 = 6$
$Q_x = 11$

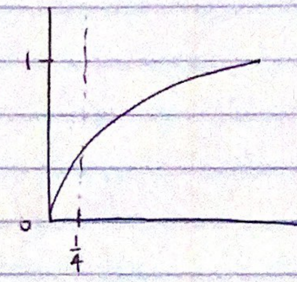


Since in this case the beginning is also the end, we can evaluate the vector-line integral to be zero.

$\boxed{0}$

2.) $\int_{\frac{1}{4}}^{1}\int_{0}^{\sqrt{x}} f(x,y)\,dy\,dx$





$y=0 \qquad x=\frac{1}{4}$

$y=\sqrt{x} \qquad x=1$

$x=y^2$

$$\boxed{\int_{0}^{1}\int_{\frac{1}{4}}^{y^2} f(x,y)\,dx\,dy}$$

3.) $2\cos(x+y)+4\cos(x+z)+8\cos(y+z)=7$

$(\frac{\pi}{6},\frac{\pi}{6},\frac{\pi}{6})$

$2\cos(x+\frac{\pi}{6})+4\cos(x+\frac{\pi}{6})+8\cos(y+\frac{\pi}{6})=7$

$2\cos(\frac{\pi}{3})+4\cos(\frac{\pi}{3})+8\cos(\frac{\pi}{3})=7$

$2\cdot\frac{1}{2}+4\cdot\frac{1}{2}+8\cdot\frac{1}{2}=7$

$1+2+4=7$ ✓

4.) $a \times b = i + j ~~+~~ - k$

$b \times c = i - j + k$

$a \times c = 2i + j + 2k$

$a_2 b_3 - a_3 b_2 = 1 \qquad a_3 b_1 - a_1 b_3 = 1 \qquad a_1 b_2 - a_2 b_1 = -1$

$b_2 c_3 - \cancel{b}a_3 c_2 = 1 \qquad b_3 c_1 - b_1 c_3 = -1 \qquad b_1 c_2 - b_2 c_1 = 1$

$a_2 c_3 - a_3 c_2 = 2 \qquad a_3 c_1 - a_1 c_3 = 1 \qquad a_1 c_2 - a_2 c_1 = 2$

$a_2 b_3 - a_3 b_2 = b_2 c_3 - b_3 c_2 \qquad a_3 c_1 - a_1 c_3 = a_3 b_1 - a_1 b_3$

$a_2 b_3 - a_3 b_2 \cancel{+} - b_2 c_3 + b_3 c_2 = 0 \qquad a_3 c_1 - a_1 c_3 - a_3 b_1 + a_1 b_3 = 0$

$b_3 (a_2 + c_2) - b_2 (a_3 + c_3) = 0 \qquad a_3 (c_1 - b_1) - a_1 (c_3 - b_3)$

if $a \times (b + c)$
$a \times b + a \times c$

$(a + b + c) \times (2a - b + 3c)$

~~aaaaaaa~~

$2a \times a - a \times b + a \times 3c + b \times 2a \cancel{-} b \times b + b \times 3c$

$+ c \times 2a - c \times b + 3c \times c$

$2 \cdot (c) = 2c$

$a \times a = 0 \qquad 0 - (i + j - k) + 3(2i + j + 2k) + 2(i + j - k)$

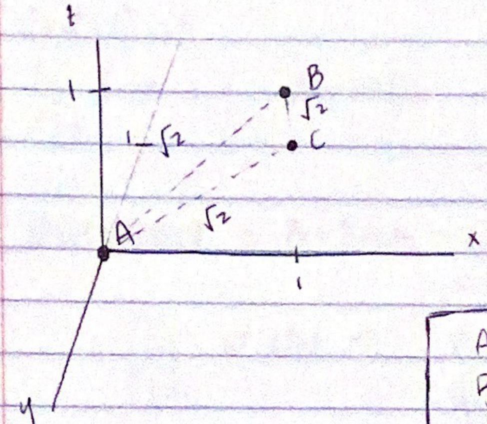
$+ 0 + 3(i - j + k) + 2(2i + j + 2k) + 3(0) =$

$0 - (i + j - k) + (6i + 3j + 6k) + (2i + 2j - 2k) + (3i - 3j + 3k) + (4i + 2j + 4k)$

$-i - j + k + 6i + 3j + 6k + 2i + 2j - 2k + 3i - 3j + 3k + 4i + 2j + 4k$

$\boxed{14i + 3j + 12k}$

5) $A = (0,0,0) \quad B = (1,0,1) \quad C = (1,1,0)$



$\overrightarrow{AB} = \langle 1, 0, 1 \rangle \quad \|\overrightarrow{AB}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$

$\overrightarrow{BC} = \langle 0, 1, -1 \rangle \quad \|\overrightarrow{BC}\| = \sqrt{0^2+1^2+(-1)^2} = \sqrt{2}$

$\overrightarrow{AC} = \langle 1, 1, 0 \rangle \quad \|\overrightarrow{AC}\| = \sqrt{1^2+1^2+0^2} = \sqrt{2}$

$\therefore$ equilateral triangle!

$\rightarrow 60°$ for all angles!

$\boxed{\begin{array}{c} A = 60° \\ B = 60° \\ C = 60° \end{array}} \rightarrow \boxed{\begin{array}{c} A = \frac{\pi}{3} \text{ rad} \\ B = \frac{\pi}{3} \text{ rad} \\ C = \frac{\pi}{3} \text{ rad} \end{array}}$

6.) $f(x,y,z) = x^3+y^3+z^3+xyz$

$(1,1,1) \rightarrow (-1,-1,-1) \quad (\langle -2,-2,-2 \rangle)$

$\left.\begin{array}{l} f_x = 3x^2+yz \\ f_y = 3y^2+xz \\ f_z = 3z^2+xy \end{array}\right\} \nabla f$

$u = \frac{v}{|v|} \rightarrow \sqrt{(-2)^2+(-2)^2+(-2)^2} = \sqrt{4+4+4} = \sqrt{12}$

$u = \left\langle -\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}} \right\rangle$

$\nabla f(1,1,1) = 3(1)^2+(1)(1),\ 3(1)^2+(1)(1) + 3(1)^2+(1)(1)$

$\nabla f(1,1,1) = 4, 4, 4$

dot product $\rightarrow \nabla f \cdot u \rightarrow (4, 4, 4) \cdot \left(-\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}}, -\frac{2}{\sqrt{12}}\right)$

$3\left(4\left(-\frac{2}{\sqrt{12}}\right)\right) = \boxed{-\frac{24}{\sqrt{12}}}$

7) $\frac{\partial g}{\partial u}$ @ $(u,v) = (0,1)$

$g = 3x^2 - 3y^2, \quad x = e^u \cos v, \quad y = e^u \sin v$

$\frac{\partial g}{\partial u} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial g}{\partial y} \cdot \frac{\partial y}{\partial u}$

$\frac{\partial g}{\partial x} = 6x \quad \frac{\partial x}{\partial u} = e^u \cos v \quad \frac{\partial g}{\partial y} = -6y \quad \frac{\partial y}{\partial u} = e^u \sin v$

$6xe^u \cos v + (-6y e^u \sin v) = 6(e^u \cos v)e^u \cos v - 6(e^u \sin v)e^u \sin v$

$6(e^0 \cos(1))(e^0 \cos(1)) - 6(e^0 \sin(1))(e^0 \sin(1)) =$

$6[(\cos(1))^2 - (\sin(1))^2] = 6[(1-\sin^2(1) - \sin^2(1))] = 6[1 - 2\sin^2(1)]$

$\boxed{\approx \text{approx. } -2.497}$

8.) $F = \langle 3x + \cos(y^3 + yz), -2y + e^{x+z^2}, 5z + \sin(xy^3 + e^x) \rangle$

$x^2 + y^2 + z^2 < 4$

$\iint_S F \cdot ds = \iiint \text{div} \cdot F \, dv$

divergence $= 3 - 2 + 5 = 6$

$\iiint 6 \, dv \rightarrow$

$x = \rho \sin\varphi \cos\theta$

$y = \rho \sin\varphi \sin\theta$

~~$z = \rho \sin\varphi$~~

$z = \rho \cos\varphi$

b/c $y < 0$!

$x^2 + y^2 + z^2 = \rho^2$

ρ 0 to 2

$z > 0$

~~$0 \le \rho \le 2$~~ $\quad 0 \le \rho \le 2$

~~$0 \le \varphi \le \pi$~~ $\quad \pi \le \varphi \le 0$

~~$0 \le \theta \le 2\pi$~~ $\quad 0 \le \theta \le \pi$

~~$\int_0^{\pi} \int_0^{2\pi} \int_0^2 6\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$~~

~~$\int_0^2 6\rho^2 \sin\varphi \, d\rho = 6 \cdot \frac{\rho^3}{3} \cdot 2\rho^3 \sin\varphi \Big|_0^2 = 16\sin\varphi$~~

~~$\int_0^{2\pi} 16\sin\varphi \, d\varphi = -16\cos\varphi \Big|_0^{2\pi}$~~

$\int_{-\pi}^{0} \int_0^{\pi} \int_0^2 6\rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta \rightarrow \int_0^2 6\rho^2 \sin\varphi \, d\rho = 2\rho^3 \sin\varphi \Big|_0^2 = 16\sin\varphi$

$\int_0^{\pi} 16\sin\varphi \, d\varphi = -16\cos\varphi \Big|_0^{\pi} = 32 \quad \int_{-\pi}^{0} 32 \, d\theta = 32\theta \Big|_{-\pi}^{0} = \boxed{32\pi}$

9.) $F = \langle 3z, 2x, y+z \rangle$ $\quad z = 2x+3y$

$$\iint -P\frac{\partial z}{\partial x} - Q\frac{\partial z}{\partial y} + R \, dA$$

$$\iint -3z(2) - 2x(3) + y+z \, dA$$

$$\iint -6z - 6x + y + z \, dA$$

$$\iint -6x + y - 5z \, dA$$

$$\iint -6x + y - 5(2x+3y) \, dA$$

$$\iint -6x + y - 10x - 15y \, dA$$

$$\int_0^1\int_0^1 -16x - 14y \, dA$$

$$\int_0^1\int_0^1 -16x - 14y \, dx\,dy$$

$$\int_0^1 -16x - 14y \, dx = -\frac{16x^2}{2} - 14xy \Big|_0^1 = -14y - 8$$

$$\int_0^1 -14y - 8 \, dy = -\frac{14y^2}{2} - 8y \Big|_0^1 = \boxed{-15}$$?? Should not be negative! (surface area)

10.) $f(x,y) = 4x - y^2 - \ln(2x+y)$

$f_x = 4 - 0 - \frac{1}{2x+y}(2) = f_x = 4 - \frac{2}{2x+y}$

$f_y = -2y - \frac{1}{2x+y}(1) = f_y = -2y - \frac{1}{2x+y}$

→ set to 0

$4 - \frac{2}{2x+y} = 2y - \frac{1}{2x+y}$

$4 = \frac{2}{2x+y} \rightarrow (2x+y = \frac{1}{2})2 \rightarrow 4x+2y = 1$

$-2y = \frac{1}{2x+y} \rightarrow -4xy - 2y^2 = 1$

$-4xy - 2y^2 = 4x + 2y$

$4x + 2y + 4xy + 2y^2 = 0$

$4x(1+y) + 2y(1+y) = 0$

$(4x+2y)(1+y) = 0$

$y = -1, \quad 4x - 2 = 0$

$x = \frac{1}{2}$

$4(\frac{1}{2}) + 2(-1) = 0$

$2 - 2 = 0$

$-4(\frac{1}{2})(-1) - 2(1) = 1$

$+2 - 2$

$4x + 2y + 4xy + 2y^2 = 0$

$4x + 4xy + 2y + 2y^2 = 0$

$4x(1+y) + 2y(1+y) = 0$

$(4x+2y)(1+y) = 0$

$2(\frac{1}{2}) + (-1) = \frac{1}{2}$

$1 - 1 = 0$

$y + 1 = 0$

$y = -1$

$4x + 2y = 0$

$4x - 2 = 0$

$x = \frac{2}{4}$ $\quad x = \frac{1}{2}$??

$x = \frac{1}{2}, \; y = -1$

$f_{yy} = \left(-2y - \frac{1}{2x+y}\right)\frac{d}{dy} = -2 + \frac{1+0}{(2x+y)^2} = \frac{1}{(2x+y)^2} - 2$

$f_{xx} = \left(4 - \frac{2}{2x+y}\right)\frac{d}{dx} = 0 + \frac{2(2(1)+0)}{(2x+y)^2} = \frac{4}{(2x+y)^2}$

$f_{xy} = 4 - \frac{2}{2x+y}$

$0 + \frac{2(1+0)}{(2x+y)^2}$

$D = \frac{4}{(2x+y)^2}\left(\frac{1}{(2x+y)^2} - 2\right) - \left(\frac{2}{(2x+y)^2}\right)^2$

$f_{xx}(\frac{1}{2},-1)\ \frac{4}{(2(\frac{1}{2})-1)^2} \quad \frac{4}{0}$

11.) $f(x,y,z) = \sqrt{2x^2+3y^2+z^2}$ $\quad \sqrt{2(1)^2+3(1)^2+(2)^2} = \sqrt{2+3+4} = \sqrt{10}$

$f_x = \frac{1}{2}(2x^2+3y^2+z^2)^{1-\frac{1}{2}} \cdot 2\,2x + 0 + 0$

$f_x = \dfrac{2x}{\sqrt{2x^2+3y^2+z^2}}$

$f_y = \dfrac{3y}{\sqrt{2x^2+3y^2+z^2}}$

$f_z = \dfrac{z}{\sqrt{2x^2+3y^2+z^2}}$

formula

$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) \cdots$

$\sqrt{10} + \frac{2}{\sqrt{10}}(x-1) + \frac{3}{\sqrt{10}}(y-1) + \frac{2}{\sqrt{10}}(z-2)$

$\sqrt{10} + \frac{2}{\sqrt{10}}x - \frac{2}{\sqrt{10}} + \frac{3}{\sqrt{10}}y - \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{10}}z - \frac{2}{\sqrt{10}}$

$L(x,y,z)\ \frac{2}{\sqrt{10}}x + \frac{3}{\sqrt{10}}y + \frac{2}{\sqrt{10}}z - \frac{1}{\sqrt{10}} + \sqrt{10} =$

$\boxed{\approx 5.6924}$

12.) $\int_0^{\sqrt{2}/2}\int_0^{x} x\,dy\,dx + \int_{\sqrt{2}/2}^{1}\int_0^{\sqrt{1-x^2}} x\,dy\,dx$

$r^2 = x^2+y^2$

$r = 0$ to 1

$x = \frac{\sqrt{2}}{2}$, $x = r\cos\theta$

$\cos\theta = \frac{\sqrt{2}}{2}$, $\cos\theta = 0$

$\boxed{\theta = \frac{\pi}{4}}$ $\quad \boxed{\theta = \frac{\pi}{2}}$

$\cos\theta = 1$

$\theta = 0$

$\int_{\pi/4}^{\pi/2}\int_0^{1} r\cos\theta(r)\,dr\,d\theta +$

$\int_0^{\pi/2}\int_0^{1} r\cos\theta(r)\,dr\,d\theta$

$\int_{\pi/4}^{\pi/2} \frac{\cos\theta}{3} = \frac{1}{3} - \frac{\sqrt{2}}{3} +$

$\int_0^{\pi/2} \frac{\sin\theta}{3} \to \frac{1}{3}$

$\frac{1}{3} - \frac{\sqrt{2}}{3} + \frac{1}{3} =$

$\boxed{\frac{2-\sqrt{2}}{3} \approx 0.431}$

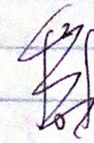
13.) $\int_0^4 \int_0^{\sqrt{4-z^2}} \int_{-\sqrt{4-z^2-y^2}}^{0} x^2 y z \, dx\,dy\,dz$

$\underline{\rho = 0 \text{ to } 2}$ (b/c in we see now $\rho^2 = x^2+y^2+z^2$ equals 2)

$-\sqrt{4-z^2-y^2} \le x \le 0 \quad \rightarrow -\frac{\pi}{2}$

$0 \le y \le \sqrt{4-z^2}$

$0 \le z \le 4$



$(\sqrt{x^2+y^2})^2 + z^2 = 4$

$z^2 + z^2 = 4$

$z^2 = 2$

$z = \sqrt{2}$

$\rho\cos\varphi = \sqrt{2}$

$2\cos\varphi = \frac{\sqrt{2}}{2}$

$\varphi = \frac{\pi}{4}$

$x = \rho\sin\varphi\cos\theta$

$y = \rho\sin\varphi\sin\theta$

$z = \rho\cos\varphi$

$\rightarrow (\rho^2 \sin^2\varphi \cos^2\theta)(\rho\sin\varphi\sin\theta)(\rho\cos\varphi)\rho^2\sin\varphi$

$\rho^6 \sin^4\varphi \cos\varphi \cos^2\theta \sin\theta \, d\rho\, d\theta\, d\varphi$

$$\boxed{\int_0^{\pi/4} \int_{-\frac{\pi}{2}}^{0} \int_0^2 \rho^6 \sin^4\varphi \cos\varphi \cos^2\theta \sin\theta \, d\rho\, d\theta\, d\varphi}$$

14.) $r(t) = \langle 5, 3\sin t, 3\cos t\rangle$ @ $t = \frac{\pi}{3}$

$\rightarrow$ mag $= \sqrt{0^2 + (3\cos t)^2 + (-3\sin t)^2} = 3$

$9(\cos^2 t + \sin^2 t)$

$r'(t) = \langle 0, 3\cos t, -3\sin t\rangle$

$r''(t) = \langle 0, -3\sin t, -3\cos t\rangle$ } cross prod.

$(3\cos t)(-3\cos t) - (-3\sin t)(-3\sin t) = -9\cos^2 t - 9\sin^2 t$

$-9(\cos^2 t + \sin^2 t) = -9, 0, 0$

mag of xprod: $\sqrt{(-9)^2 + 0^2 + 0^2} = \sqrt{81} = 9$

$\frac{\|\text{xprod}\|}{\|r'(t)\|^3} = \frac{9}{3^3} = \frac{9}{27} = \boxed{\frac{1}{3}}$ $\rightarrow$ constant?

15.) $r(u,v) = \langle u^2, 2uv, v^2\rangle$

$x = u^2$
$y = 2uv$
$z = v^2$

$0 < u < v$
$u < v < 1$

$0 \leq \sqrt{x} \leq 1$

$r_u = \langle 2u, v, 0\rangle$ $\quad r_u \times r_v = 2v^2, -4uv, 2u^2$

$r_v = \langle 0, u, 2v\rangle$

$$\int_0^1 \int_0^v f(x,y,z) \cdot (2v^2, -4uv, 2u^2)\, dA$$

16.) $f(x,y,z) = xy^2z^3$ @ pt $(1,1,1)$

$g(x,y,z) = x + y^2 + z^3$

$\nabla f \langle y^2z^3,\ 2xyz^3,\ 3xyz^2 \rangle$

$\nabla g \langle 1,\ 2y,\ 3z^2 \rangle$

$$\nabla f \circ \nabla g : \langle y^2z^3, 2xyz^3, 3xyz^2 \rangle \circ \langle 1, 2y, 3z^2 \rangle$$

$$(1,1,1)(y^2z^3,\ 4xy^2z^3,\ 9xyz^4) \to ((1)^2(1)^2,\ 4(1)(1)^2(1)^3,\ 9(1)(1)(1)(1)^4)$$

$$\boxed{(1, 4, 9)}$$

17.) $\displaystyle \lim_{(x,y,z,w)\to(0,0,0,0)} \frac{(x+y)^2 - (z+w)^2}{x+y-z-w} = \frac{0}{0}$

@ $(x+1, 0, 0, 0)$ $\quad \dfrac{(x+1+0)^2 - (0+0)^2}{(x+1)+0-0-0} = \dfrac{(x+1)^2}{x+1} = x+1$

@ $(0, y+1, 0, 0)$ $\quad \dfrac{(0+y+1)^2 - (0+0)^2}{0+(y+1)-0-0} = \dfrac{(y+1)^2}{y+1} = y+1$

@ $(0, 0, z+1, 0)$ $\quad \dfrac{(0+0)^2 - (z+1+0)^2}{0+0-(z+1)-0} = \dfrac{(z+1)^2}{-z+1} = z+1$

@ $(0, 0, 0, w+1)$ $\quad \dfrac{(0+0)^2 - (0+w+1)^2}{0+0-0-(w+1)} = \dfrac{-(w+1)^2}{w+1} = w+1$

~~The limit DNE~~ when we tried to plug in our given pt, we got $\frac{0}{0}$. since that means it is indeterminate, we tried to find functions that passed through. we found that we always ended w/ some $t+1$. let t go to zero for all, and we get 1.

$\boxed{\text{the limit is } 1}$

16. (1, 4, 9)

17. limit is 1.

Sign the following declaration:

I Hereby declare that all the work was done by myself. I was allowed to use Maple (unless specifically told not to), calculators, the book, and all the material in the web-page of this class but **not** other resources on the internet.

I only spent (at most) 3 hours on doing the exam. The last 30 minutes were spent in checking and double-checking the answers.

I also understand that I may be subject to a random short chat to verify that I actually did it all by myself.

Signed: Niharika Kompella

Possibly useful facts from school Geometry (that you are welcome to use): (i) The area of a circle radius r is πr^2. (ii) The circumference of a circle radius r is $2\pi r$ (iii) The parametric equation of an ellipse with axes a b and parallel to the x and y axes respectively is $x = a\cos\theta$, $y = b\cos\theta$, $0 < \theta < 2\pi$. (iv) The area of an ellipse with axes a and b is πab (v) The volume and surface area of a sphere radius R are $\frac{4}{3}\pi R^3$ and $4\pi R^2$ respectively (vi) The volume of a cone is the area of the base times the height over 3. (vii) The volume of a pyramid is the area of the base times the height over 3. (viii) The area of a triangle is base times height over 2.

Formula that you may (or may not) need

If the surface S is given in **explicit** notation $z = g(x,y)$, above the region of the xy-plane , D, then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\iint_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA \ .$$

1. (12 pts.) **Without using Maple (or any software)** Compute the **vector-field line integral**

$$\int_C (\cos(e^{\sin x}) + 5y)dx + (\sin(e^{\cos y}) + 11x)dy \quad ,$$

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SSC: (circle) None / I / II / I and II

MATH 251 (22,23,24) [Fall 2020], Dr. Z. , Final Exam , Tue., Dec. 15, 2020

Email the completed test, renamed as finalFirstLast.pdf to DrZcalc3@gmail.com no later than 3:30pm, (or, in case of conflict, three and half hours after the start).

WRITE YOUR FINAL ANSWERS BELOW

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1. 0

2.0<y<1, ¼<x<y^2

3.

4.14i+3j+12k

5.all three angles are pi/3 rad

6.-24/sqrt(12

7.approx. -2.497

8. 32pi

9. 15

10.

11. approx. 5.6924

12. approx. 0.431

13. rho^6 sin^4phi cosphi cos^2theta sintheta drho dtheta dphi from 0 to 2, -pi/2 to 0, and 0 to pi/4

14. 1/3

15. f.<2v^2, -4uv, 2u^2> du dv from 0 to 0, 0 to 1