Solutions to the "QUIZ" for Nov. 9, 2009

1. Determine whether or not the vector field

$$F(x,y,z) = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$$

is conservative. If it is conservative, find a function f such that $\mathbf{F} = \nabla f$.

Sol. Here $\mathbf{F} = \langle y^2z^3\,,\,2xyz^3\,,\,3xy^2z^2\rangle$,

so $F_1 = y^2z^3$, $F_2 = 2xyz^3$, $F_3 = 3xy^2z^2$.

$\frac{\partial F_1}{\partial y} = 2yz^3$, $\frac{\partial F_2}{\partial x} = 2yz^3$,

so $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ and the first condition is OK. Also

$\frac{\partial F_1}{\partial z} = 3y^2z^2$, $\frac{\partial F_3}{\partial x} = 2y^2z^2$,

so $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ and the second condition is OK. Finally

$\frac{\partial F_2}{\partial z} = 6xyz^2$, $\frac{\partial F_3}{\partial y} = 6xyz^2$,

so $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$ and the third condition is OK.

Now it is time to find the potential function f .

From

$$\frac{\partial f}{\partial x} = F_1 \quad ,$$

We get

$$\frac{\partial f}{\partial x} = y^2z^3 \quad .$$

Integrating with respect to x, we get

$$f(x,y,z) = \int y^2z^3\,dx = xy^2z^3 + g(y,z) \quad .$$

Using

$$\frac{\partial f}{\partial y} = F_2 \quad ,$$

in other word

$$\frac{\partial f}{\partial y} = 2xyz^3 \quad ,$$

We get:

$$2xyz^3 + g_y(y,z) = 2xyz^3 \quad .$$

This meant

$$g_y(y,z) = 0 \quad .$$

Integrating with respect to y, we get:

$$g(y,z) = \int 0\,dy = 0 + h(z) \quad .$$

Going back above, we have:

$$f(x,y,z) = xy^2z^3 + h(z) \quad .$$

Using

$$\frac{\partial f}{\partial z} = F_3 \quad ,$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2 \quad ,$$

we get

$$3xy^2z^2 + h'(z) = 3xy^2z^2$$

Doing the algebra, we get

$$h'(z) = 0 \quad .$$

Integrating, we get

$$h(z) = \int 0\,dz = 0 + C$$

So

$$f(x,y,z) = xy^2z^3 + C \quad .$$

But you don't have to write the $+C$.

Ans. $f(x,y,z) = xy^2z^3$ (type: multivariable function).

Comments: 1. About %50 got it perfectly. Most people knew how to do it. 2. This particular problem could be done "by inspection", but you are still expected to follow the general method.

2. Show that the line integral

$$\int_C 2xy^2\,dx \,+\, 2x^2y\,dy \quad ,$$

is independent of the path C, and evaluate it if C is *any* path from $(1,0)$ to $(0,1)$.

Sol. Here $\mathbf{F} = \langle\, 2xy^2\,, 2x^2y\,\rangle$. So $F_1 = 2xy^2, F_2 = 2x^2y$. Since $(F_1)_y = 4xy, (F_2)_x = 4xy$, $(F_1)_y = (F_2)_x$ so this is a **conservative** vector field.

Let's find a potential function.

$$f(x,y) = \int 2xy^2\,dx = x^2y^2 + g(y) \quad ,$$

So

$$\frac{\partial f}{\partial y} = 2x^2y + g'(y) \quad .$$

Using

$$\frac{\partial f}{\partial y} = 2x^2 y \quad .$$

We get

$$2x^2 y + g'(y) = 2x^2 y$$

So $g'(y) = 0$, and so $g(y) = C$, and

$$f(x, y) = x^2 y^2 \quad .$$

The line-integral (regadless of the path) is always

$$f(END) - f(START) \quad .$$

The question tells you that the path starts at $(1, 0)$ and ends at $(0, 1)$. So the answer is:

$$f(0, 1) - f(1, 0) = 0^2 \cdot 1 - 1^2 \cdot 0 = 0 \quad .$$

Ans. 0 (type number).

Comments: About %40 got it, but many people ran out of time.