Solutions to MATH 251 (4-6,11), Dr. Z.'s , Exam 1

1. (10 points) Use the chain rule to find f_u and f_v if

$$f(x,y) = x^3 + y^3 \quad , \quad x = e^{u+v} \quad , \quad y = 2u + 3v \quad .$$

Express your answer in terms of u and v.

Sol.

The **types** of the answer are: functions of u and v.

The chain rule (for two variables) says

$$f_u = (f_x)(x_u) + (f_y)(y_u) \quad , \quad f_v = (f_x)(x_v) + (f_y)(y_v) \quad .$$

So we have

$$f_u = (f_x)(x_u) + (f_y)(y_u) = (3x^2)(e^{u+v}) + (3y^2)(2) = 3x^2 e^{u+v} + 6y^2 \quad ,$$

$$f_v = (f_x)(x_v) + (f_y)(y_v) = 3x^2 e^{u+v} + 9y^2 \quad .$$

Finally, replacing x and y by their expressions in (u, v), we get

$$f_u = 3(e^{u+v})^2 e^{u+v} + 6(2u+3v)^2 = 3e^{3u+3v} + 6(2u+3v)^2 \quad ,$$

$$f_v = 3(e^{u+v})^2 e^{u+v} + 9(2u+3v)^2 = 3e^{3u+3v} + 9(2u+3v)^2 \quad .$$

Ans. $f_u = 3e^{3u+3v} + 6(2u+3v)^2$, $f_v = 3e^{3u+3v} + 9(2u+3v)^2$.

2. (10 points) Find an equation of the tangent plane to the surface

$$xz + 2x^2y + y^2z^3 = 11$$

at the point $(2, 1, 1)$.

Sol.

The **type** of the answer is: Equation of a plane.

Since this surface is given in **implicit format** $F(x, y, z) = C$, where $F(x, y, z)$ is a function of (x, y, z) (not to be confused with the **explicit format** $z = f(x, y)$), the relevant formula is

$$\bigtriangledown F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0 \quad ,$$

where $\bigtriangledown F$ is the gradient at the specified point.
First, let's make sure that the problem makes sense, by plugging $x = 2, y = 1, z = 1$ into the equation getting:

$$2 \cdot 1 + 2 \cdot 2^2 \cdot 1 + 1^2 \cdot 1^3 = 11 \quad ,$$

since this is correct, we must go on. Our $F(x, y, z)$ is $xz + 2x^2y + y^2z^3 - 11$, so We have

$$\bigtriangledown F = \langle F_x, F_y, F_z \rangle = \langle z + 4xy, 2x^2 + 2yz^3, x + 3y^2z^2 \rangle \quad .$$

At the specified point $(x, y, z) = (2, 1, 1)$ this is

$$\bigtriangledown F = \langle 1 + 4(2)(1), 2(2)^2 + 2(1)(1)^3, 2 + 3(1)^2(1)^2 \rangle = \langle 9, 10, 5 \rangle$$

Plugging it into the general formula we get

$$\langle 9, 10, 5 \rangle \cdot \langle x - 2, y - 1, z - 1 \rangle = 0 \quad ,$$

that spells out to

$$9(x - 2) + 10(y - 1) + 5(z - 1) = 0$$

This is OK for the final answer, but it is even better to further simplify:

$$9x + 10y + 5z = 33 \quad .$$

Ans. : $9x + 10y + 5z = 33$.

Comment: it is crucial to plug-in the point into $\bigtriangledown F$. If you don't you get a **monster**, that is not a plane at all, and you would get zero points.

3. (10 points) Find the maximal rate of change of $f(x,y,z) = x^2y^3z^4$ at $(1,1,1)$, and the **unit** direction where it occurs.

Sol.

The **types** of the answers are: number and vector of numbers.

$$\bigtriangledown f = \langle 2xy^3x^4, 3x^2y^2z^4, 4x^2y^3z^3 \rangle$$

At the specified point $(1,1,1)$ we have

$$\bigtriangledown f = \langle 2,3,4 \rangle$$

The *maximum rate of change* is the **magnitude**

$$|\bigtriangledown f| = |\langle 2,3,4 \rangle| = \sqrt{2^2+3^2+4^2} = \sqrt{4+9+16} = \sqrt{29} \quad .$$

The **unit** direction is the gradient divided by its length

$$\frac{\langle 2,3,4 \rangle}{|\langle 2,3,4 \rangle|} = \frac{\langle 2,3,4 \rangle}{\sqrt{29}} = \langle \frac{2}{\sqrt{29}}, \frac{3}{\sqrt{29}}, \frac{4}{\sqrt{29}} \rangle \quad .$$

4. (10 points) Compute f_{xx}, f_{xy}, and f_{yy} if

$$f(x,y) = e^{x^2+y^2} \quad .$$

Sol.

The **type** of the answers is: functions of x, y.

We first need the *first derivatives*:

$$f_x = 2xe^{x^2+y^2}$$

$$f_y = 2ye^{x^2+y^2}$$

Now we can go on to the *second derivatives.* We need the **product rule**:

$$f_{xx} = (2xe^{x^2+y^2})' = (2x)'e^{x^2+y^2} + 2x(e^{x^2+y^2})'$$

(where at the present context $'$ means differentiation with respect to x) . This becomes

$$(2x)'e^{x^2+y^2} + 2x(e^{x^2+y^2})' = 2e^{x^2+y^2} + 2x(e^{x^2+y^2})(2x) = 2e^{x^2+y^2} + 4x^2e^{x^2+y^2} = (2+4x^2)e^{x^2+y^2} \quad .$$

To get f_{xy} is a bit easier

$$f_{xy} = \frac{\partial}{\partial y}(2xe^{x^2+y^2}) = 2xe^{x^2+y^2}(2y) = 4xye^{x^2+y^2} \quad .$$

Finally f_{yy} is analogous to f_{xx} getting

$$f_{yy} = (2+4y^2)e^{x^2+y^2} \quad .$$

Ans.: $f_{xx} = (2+4x^2)e^{x^2+y^2}$, $f_{xy} = 4xye^{x^2+y^2}$, $f_{yy} = (2+4y^2)e^{x^2+y^2}$.

Comment: Some people need to review the product rule and the chain rule for single-variable functions.

5. (10 points) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if

$$z^3 + x^3 + y^3 + 3xyz = 6 \quad .$$

Sol.

The **types** of the answer are: functions of x, y (but also involving the implicilty-defined function z)

The easiest way is to use the ready-made formulas

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} \quad .$$

In this problem $F(x, y, z) = z^3 + x^3 + y^3 - 3xyz - 6$. So we have

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 + 3yz}{3z^2 + 3xy} = -\frac{3(x^2 + yz)}{3(z^2 + xy)} = -\frac{x^2 + yz}{z^2 + xy} \quad ,$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3y^2 + 3xz}{3z^2 + 3xy} = -\frac{3(y^2 + xz)}{3(z^2 + xy)} = -\frac{y^2 + xz}{z^2 + xy} \quad .$$

Ans.: $\frac{\partial z}{\partial x} = -\frac{x^2+yz}{z^2+xy}$, $\frac{\partial z}{\partial y} = -\frac{y^2+xz}{z^2+xy}$.

6. (10 points) Find a parametric equation of the line of intersection of the planes $x+2y+3z=6$ and $3x+2y+z=6$.

Sol.

The **type** of the answer is: parametric equation of a line.

We need a **direction** and (any!) **point**.
The normal of the first plane is $\langle 1,2,3\rangle$, the normal of the second plane is $\langle 3,2,1\rangle$.
Since our desired line lies on **both** planes, it is perpendicular to both these vectors, so its direction is the same direction as the **cross product**.

$$\langle 1,2,3\rangle \times \langle 3,2,1\rangle =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{vmatrix} \quad =$$

$$\mathbf{i}\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 3 \\ 3 & -1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & 2 \\ 3 & 2 \end{vmatrix}$$

$$= \mathbf{i}(-4) + \mathbf{j}8 + \mathbf{k}(-4) = \langle -4, 8, -4\rangle \quad .$$

Now we still need a **point**. Quite arbitrarily let's pick $y=1$ and plug it into the two planes

$$x+2+3z=6 \quad , \quad 3x+2+z=6 \quad ,$$

yielding:

$$x+3z=4 \quad , \quad 3x+z=4 \quad .$$

Solving these equations gives $x=1, z=1$, yielding the point $(1,1,1)$.
So the equation of the common line is:

$$\langle 1,1,1\rangle + t\langle -4,8,-4\rangle \quad (-\infty < t < \infty) \quad .$$

and in *scalar form* it is:

$$x = 1-4t \quad , \quad y = 1+8t \quad , \quad z = 1-4t \quad (-\infty < t < \infty) \quad .$$

Ans. $x = 1-4t \quad , \quad y = 1+8t \quad , \quad z = 1-4t \quad (-\infty < t < \infty)$

Comments: There are many are right answers, since there many choices of points. Also since $-\infty < t < \infty$ we can simplify further and divide the direction by 4, so an even nicer answer is $x = 1-t \quad , \quad y = 1+2t \quad , \quad z = 1-t \quad (-\infty < t < \infty)$

7. (10 points) A certain particle has law of motion

$$\mathbf{r}(t) = \langle \sin t \, , \, \cos 2t \, , \, e^t \, \rangle \quad .$$

Find its velocity, acceleration, and speed at $t = \pi/6$.

Sol.

The **types** of the answers are: two vectors of numbers and a number.

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle \cos t \, , \, -2\sin 2t \, , \, e^t \, \rangle \quad ,$$

$$\mathbf{a}(t) = \mathbf{v}'(t) = \langle -\sin t \, , \, -4\cos 2t \, , \, e^t \, \rangle \quad ,$$

Now it is time to plug-in $t = \pi/6$, we have

$$\mathbf{v}(\pi/6) = \langle \cos \pi/6 \, , \, -2\sin \pi/3 \, , \, e^{\pi/6} \, \rangle = \langle \frac{\sqrt{3}}{2} \, , -2\frac{\sqrt{3}}{2}, \, , \, e^{\pi/6} \, \rangle = \langle \frac{\sqrt{3}}{2} \, , -\sqrt{3}, \, , \, e^{\pi/6} \, \rangle$$

$$\mathbf{a}(\pi/6) = \langle -\sin \pi/6 \, , \, -4\cos \pi/3 \, , \, e^{\pi/6} \, \rangle = \langle -\frac{1}{2} \, , \, -4 \cdot \frac{1}{2} \, , \, e^{\pi/6} \, \rangle = \langle -\frac{1}{2} \, , \, -2 \, , \, e^{\pi/6} \, \rangle \quad .$$

Finally the speed at $t = \pi/6$ is:

$$|\mathbf{v}(\pi/6)| = |\langle \frac{\sqrt{3}}{2} \, , -\sqrt{3}, \, , \, e^{\pi/6} \rangle| = \sqrt{(\frac{\sqrt{3}}{2})^2 + (-\sqrt{3})^2 + (e^{\pi/6})^2} = \sqrt{\frac{3}{4} + 3 + (e^{\pi/6})^2} = \sqrt{\frac{15}{4} + e^{\pi/3}}$$

Ans.: The velocity at $t = \pi/6$ is $\langle \frac{\sqrt{3}}{2} \, , -\sqrt{3}, \, , \, e^{\pi/6} \, \rangle$ and the acceleration then is $\langle -\frac{1}{2} \, , \, -2 \, , \, e^{\pi/6} \, \rangle$, while the speed then is $\sqrt{\frac{15}{4} + e^{\pi/3}}$.

Comments: 1. Some people interpreted the phrase "velocity, acceleration, and speed at $t = \pi/6$" as "velocity (at general time t), acceleration (at general time t), and speed at $t = \pi/6$". Strictly speaking, this is also a valid interpreation, and they got full credit. 2. People who didn't evaluate $\cos \pi/6$ etc. got 9 out of 10 points.

8. (10 points) Write a definite integral that describes the length of the curve

$$\mathbf{r}(t) = \langle \sin 5t\,,\, e^{2t}\,,\, e^{3t}\,\rangle \quad , \quad 0 \le t \le \pi \quad .$$

Do not try to evaluate the integral!

Sol.

The **type** of the answer is: number given in terms of a definite integral.

The relevant formula is

$$Length = \int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt \quad .$$

In this problem

$$\mathbf{r}'(t) = \langle\, 5\cos 5t\,,\, 2e^{2t}\,,\, 3e^{3t}\,\rangle$$

Taking the **magnitude** we have

$$|\mathbf{r}'(t)| = |\langle\, 5\cos 5t\,,\, 2e^{2t}\,,\, 3e^{3t}\,\rangle| = \sqrt{(5\cos 5t)^2 + (2e^{2t})^2 + (3e^{3t})^2} = \sqrt{25\cos^2 5t + 4e^{4t} + 9e^{6t}}$$

Putting it in the formula for the length we have:

$$Length = \int_0^{\pi} \sqrt{25\cos^2 5t + 4e^{4t} + 9e^{6t}}\, dt$$

That's it! (don't try to "simplify", or you will lose lots of points, perhaps even all of them, if your "simplifications" are wrong.)

Ans.: the length is $\int_0^{\pi} \sqrt{25\cos^2 5t + 4e^{4t} + 9e^{6t}}\, dt$

9. (10 points, altogether) Do the following limits exist? If they do, find them. Explain!

a. (5 points) If

$$\lim_{(x,y,z)\to(1,2,3)} f(x,y,z) = 1 \quad , \qquad \lim_{(x,y,z)\to(1,2,3)} g(x,y,z) = 2$$

compute

$$\lim_{(x,y,z)\to(1,2,3)} (f(x,y,z)+g(x,y,z))^3 e^{g(x,y,z)}$$

Sol.

The **type** of the answer is: number.

$$\lim_{(x,y,z)\to(1,2,3)} (f(x,y,z)+g(x,y,z))^3 e^{g(x,y,z)} = (1+2)^3 e^2 = 27e^2 \quad .$$

Ans.: The limit exists and equals $27e^2$.

b. (5 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^4+y^4}{2x^4+y^4} \quad .$$

Sol.: The **type** of the answer is: DNE.

First Way: The limit, **only** going along the y axis is:

$$\lim_{(x,y)\to(0,0),x=0} \frac{y^4}{y^4} = \lim_{y\to 0} 1 = 1 \quad ,$$

The limit, **only** going along the x axis is:

$$\lim_{(x,y)\to(0,0),y=0} \frac{x^4}{2x^4} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2} \quad .$$

Since these are **different** the limit does not exist.

Second Way: Let's invesitage the limit along a line of general slope m, passing through the origin, given by $y = mx$:

$$\lim_{(x,y)\to(0,0),y=mx} \frac{x^4+y^4}{2x^4+y^4} = \lim_{x\to 0} \frac{x^4+(mx)^4}{2x^4+(mx)^4} = \lim_{x\to 0} \frac{x^4(1+m^4)}{x^4(2+m^4)} = \lim_{x\to 0} \frac{1+m^4}{2+m^4} = \frac{1+m^4}{2+m^4} \quad .$$

Since this **depends** on m, the limit **does not exist**.

Ans.: DNE

10. (10 pts.) A force with magnitude $500N$ is moving a body of mass $5kg$ in the direction $\langle 3,4,0\rangle$. If at $t=0$ the body is at location $(1,2,3)$ and it is moving with velocity $\langle 2,1,4\rangle$, (i)find its position vector $\mathbf{r}(t)$ at time t ;
(ii) find its speed at time t.

Sol.

The **types** of the answers are: (i) vector of functions of t (ii) function of t .

By Newton's Second Law the **magnitude** of the acceleration is $500/5 = 100$. The direction is $\langle 3,4,0\rangle$. But we need the **unit direction**. Since $|\langle 3,4,0\rangle| = \sqrt{3^2+4^2+0^2} = \sqrt{25} = 5$, the **unit direction is**

$$\frac{\langle 3,4,0\rangle}{5} = \langle \frac{3}{5}, \frac{4}{5}, 0\rangle \quad .$$

Multiplying the magnitude of the acceleration by the unit direction gives the **acceleration vector**:

$$\mathbf{a} = 100\langle \frac{3}{5}, \frac{4}{5}, 0\rangle = \langle 60,80,0\rangle \quad .$$

To get $\mathbf{v}(t)$ we integrate

$$\mathbf{v}(t) = \int \langle 60,80,0\rangle \, dt = \langle 60t, 80t, 0\rangle + \mathbf{C}$$

Plugging-in $t=0$ gives

$$\mathbf{v}(0) = \mathbf{C} \quad .$$

From the data of the problem $\mathbf{v}(0) = \langle 2,1,4\rangle$, so $\mathbf{C} = \langle 2,1,4\rangle$, and we get:

$$\mathbf{v}(t) = \int \langle 60,80,0\rangle \, dt = \langle 60t, 80t, 0\rangle + \langle 2,1,4\rangle = \langle 60t+2, 80t+1, 4\rangle$$

To get the **position** vector, we integrate once again

$$\mathbf{r}(t) = \int \langle 60t+2, 80t+1, 4\rangle \, dt = \langle 30t^2+2t, 40t^2+t, 4t\rangle + \mathbf{C}$$

Plugging-in $t=0$ gives $\mathbf{C} = \langle 1,2,3\rangle$. Putting it above, we get

$$\mathbf{r}(t) = \langle 30t^2+2t, 40t^2+t, 4t\rangle + \langle 1,2,3\rangle = \langle 30t^2+2t+1, 40t^2+t+2, 4t+3\rangle \quad .$$

This is the answer to part (i). For part (ii) we take the magnitude of $\mathbf{v}(t)$.

$$\mathit{Speed} = |\mathbf{v}(t)| = |\langle 60t+2, 80t+1, 4\rangle| = \sqrt{(60t+2)^2+(80t+1)^2+4^2} = \sqrt{(60t+2)^2+(80t+1)^2+16}$$

This is acceptable (no need to expand the inside of the square-root).

Ans.: (i) $\langle 30t^2+2t+1, 40t^2+t+2, 4t+3\rangle$ (ii) $\sqrt{(60t+2)^2+(80t+1)^2+16}$.