MATH 251 (1-3,10 ), Dr. Z. , Solutions to Exam 1 , Thurs., Oct. 8, 2009, SEC 118

**1**. (10 pts.) Find the curvature of the curve

$$\mathbf{r}(t) = \langle e^t, \sin t, \cos t \rangle$$

at the point $(1, 0, 1)$.

**Sol.**:
The type of the answer is **number**. Since it is at a specific point.

The formula is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

We have

$$\mathbf{r}'(t) = \langle e^t, \cos t, -\sin t \rangle$$

$$\mathbf{r}''(t) = \langle e^t, -\sin t, -\cos t \rangle$$

The next thing to worry about is: **What time is it?**, in other words what is $t$ when the particle is at $(1, 0, 1)$. We need $e^t = 1$, to $t = 0$. To make sure we plug it into $\mathbf{r}(t)$ and we get

$$\mathbf{r}(t) = \langle e^0, \sin 0, \cos 0 \rangle = \langle 1, 0, 1 \rangle \quad ,$$

so $t$ is indeed 0 right now.
**Now** that we know that $t = 0$, we plug-in and get

$$\mathbf{r}'(0) = \langle e^0, \cos 0, -\sin 0 \rangle = \langle 1, 1, 0 \rangle$$

$$\mathbf{r}''(0) = \langle e^0, -\sin 0, -\cos 0 \rangle = \langle 1, 0, -1 \rangle$$

$$\mathbf{r}'(0) \times \mathbf{r}''(0) =$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} \quad =$$

$$\mathbf{i} \begin{vmatrix} 1 & 0 \\ 0 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= -\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle -1, 1, -1 \rangle \quad .$$

So $|\mathbf{r}'(0) \times \mathbf{r}''(0)| = |\langle -1, 1, -1 \rangle| = \sqrt{(-1)^2 + 1^2 + (-1)^2} = \sqrt{3}$. Also $|\mathbf{r}'(0)| = |\langle 1, 1, 0 \rangle| = \sqrt{(1)^2 + 1^2 + 0^2} = \sqrt{2}$. So $\kappa(0) = \frac{\sqrt{3}}{(\sqrt{2})^3} = \frac{\sqrt{6}}{4}$.
**Ans.** $\frac{\sqrt{6}}{4}$.
**Comments**: 1. Some people found the general $\kappa(t)$ and only at the end plugged-in $t = 0$. This is correct, but takes much longer. You should plug-in as soon as you can.
2. Some people didn't get that $t = 0$.

**2.** (10 points) Find $\frac{\partial h}{\partial q}$ at $(q,r)=(3,2)$ where $h(u,v)=ue^v$, $u=q^3$, $v=qr^2$

**Sol.**:
The **type** of the answer is: **number**.

$h_q = (h_u)(u_q) + (h_v)(v_q) = (e^v)(3q^2) + (ue^v)(r^2) = e^{qr^2}(3q^2) + q^3 e^{qr^2} r^2 = e^{qr^2}(3q^2 + q^3 r^2)$

Now we plug-in $q=3, r=2$ to get

$$e^{3\cdot 2^2}(3\cdot 3^2 + 3^3 \cdot 2^2) = 135e^{12} \quad .$$

**Ans.**: $135e^{12}$.

**3.** (10 points) Find the directional derivative of $f(x, y, z) = xy + z^3$ at $P = (3, -2, -1)$ in the direction pointing to the point $Q = (2, -3, -3)$.

**Sol.**
The **type** of the answer is: **number**.

$$\bigtriangledown f = \langle f_x, f_y, f_z \rangle = \langle y, x, 3z^2 \rangle \quad .$$

At the designated point (the point $(3, -2, -1)$) it is

$$\bigtriangledown f = \langle f_x, f_y, f_z \rangle = \langle -2, 3, 3 \rangle \quad .$$

The direction is

$$\mathbf{PQ} = \langle 2, -3, -3 \rangle - \langle 3, -2, -1 \rangle = \langle -1, -1, -2 \rangle$$

and the **unit directio** is $\mathbf{u} = \langle -1, -1, -2 \rangle / |\langle -1, -1, -2 \rangle| = \frac{\langle -1,-1,-2 \rangle}{\sqrt{6}}$.
The directional derivative is

$$\bigtriangledown f \cdot \mathbf{u} = \langle -2, 3, 3 \rangle \cdot \frac{\langle -1, -1, -2 \rangle}{\sqrt{6}} = \frac{(-2)(-1) + (3)(-1) + 3(-2)}{\sqrt{6}} = \frac{-7}{\sqrt{6}} = \frac{-7\sqrt{6}}{6} \quad .$$

**Ans.**: $\frac{-7\sqrt{6}}{6}$.

**4.** (10 points) Find an equation of the tangent plane at the given point

$$H(s,t) = te^{s^2 t} \quad ,(0,0) \quad .$$

**Sol.**:
The **type** of the answer is: Equation of a plane.

The relevant formula is

$$z - z_0 = H_s(s_0,t_0)(s-s_0) + H_t(s_0,t_0)(t-t_0) \quad .$$

At $(0,0)$, $z_0 = H(0,0) = 0e^0 = 0$.
We have:

$$H_s(s,t) = te^{s^2t}(2st) = 2st^2e^{s^2t} \quad ,$$

$$H_t(s,t) = e^{s^2t} + st^2e^{s^2t} = (1+st^2)e^{s^2t} \quad .$$

At $(0,0)$ we have:
$H_s(0,0) = 0e^{0^2\cdot 0}(0) = 0$,
$H_t(0,0) = (1 + 0 \cdot 0^2)e^{0^2 \cdot 0} = 1$.
So the equation is:

$$z - 0 = 0(s-0) + 1(t-0)$$

that becomes

$$z = t \quad .$$

**Ans.**: $z = t$, or $H = t$.

**Comments**: The traditional variables are $x, y$, so people who wrote $z = y$ got 9.5 out of 10.

**5.** (10 points) Compute $f_{yy}(2,3)$ if $f(x,y) = x\ln(y^2)$.

**Sol.**
The **type** of the answer is: Number.

Before you differentiate, **simplify!**. Since $\ln y^2 = 2\ln y$, by the ln rules,

$$f(x,y) = 2x\ln y \quad .$$

So

$$f_y(x,y) = \frac{2x}{y} = 2xy^{-1} \quad ,$$

and

$$f_{yy}(x,y) = 2x(-1)y^{-2} = \frac{-2x}{y^2} \quad .$$

Finally, plugging in $x = 2, y = 3$, yields

$$f_{yy}(2,3) = \frac{-2\cdot 2}{3^2} = -\frac{4}{9} \quad .$$

**Ans.**: $-\frac{4}{9}$.
**Comments**: Most people didn't do the first simplification, but instead did

$$f_y(x,y) = \frac{(xy^2)'}{y^2} = \frac{2yx}{y^2}$$

(here $'$ means diff. w.r.t. $y$). That's OK, it is only slighlty more complicated, if they immediately simplify this to

$$\frac{2x}{y}$$

and proceed as before. But quite a few people didn'd do the algebraic simplification and used the quotient rule for $\frac{2yx}{y^2}$, and some even did it correctly, but it took them longer than necessary.

**6.** (10 points) Use the linearization of $f(x,y)=\sqrt{x+2y}$ to approximate $f(3.01,2.96)$.

**Sol.**
The **type** of the answer is: Number.

The relevant formula is

$$L(x,y)=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0) \quad ,$$

where $(x_0,y_0)$ is "nice". Here both 3.01 and 2.96 are close to 3, so $(x_0,y_0)=(3,3)$.
We have

$$f(x,y)=(x+2y)^{1/2} \quad ,$$

So

$$f_x(x,y)=(1/2)(x+2y)^{-1/2}\cdot 1=\frac{1}{2\sqrt{x+2y}} \quad ,$$

$$f_y(x,y)=(1/2)(x+2y)^{-1/2}\cdot 2=\frac{1}{\sqrt{x+2y}} \quad ,$$

So

$$f_x(3,3)=\frac{1}{2\sqrt{3+2\cdot 3}}=\frac{1}{6} \quad ,$$

$$f_y(3,3)=\frac{1}{\sqrt{3+2\cdot 3}}=\frac{1}{3} \quad ,$$

Also

$$f(3,3)=\sqrt{3+2\cdot 3}=\sqrt{9}=3 \quad ,$$

So

$$L(x,y)=3+\frac{1}{6}(x-3)+\frac{1}{3}(y-3) \quad .$$

Finally, plugging in the actual arguments $x=3.01, y=2.96$, we get

$$L(3.01,2.96)=3+\frac{1}{6}(3.01-3)+\frac{1}{3}(2.96-3)=3+\frac{1}{600}-\frac{4}{300}=3-\frac{7}{600}=\frac{1793}{600} \quad .$$

**Ans.** $\frac{1793}{600}$.

**7.** (10 points, altogether) Do the following limits exist? If they do, find them. Explain!

**a.** (4 points)

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{1+x+y+z}$$

**Sol.**: The **type** of the answer is **number** .

The first thing to try is to **plug-it in**:

$$\lim_{(x,y,z)\to(0,0,0)} \frac{x^3+y^3+z^3}{1+x+y+z} = \frac{0^3+0^3+0^3}{1+0+0+0} = \frac{0}{1} = 0 \quad .$$

This makes perfect sense (0 on the top is no problem!), so the answer is that the limit exists and is 0.

**Ans.**: The limit exists and is 0.

**b.** (6 points)

$$\lim_{(x,y)\to(0,0)} \frac{x^2+2y^2}{x^2+3y^2} \quad .$$

**Sol.**

The **type** of the answer is **DNE** .

**First way**: Let's see what is going on when we go to the origin along the $y$ axis:

$$\lim_{(x,y)\to(0,0),x=0} \frac{2y^2}{3y^2} = \lim_{y\to 0} \frac{2}{3} = \frac{2}{3}$$

And along the $x$ axis:

$$\lim_{(x,y)\to(0,0),y=0} \frac{x^2}{x^2} = \lim_{x\to 0} \frac{1}{1} = 1$$

Since we got two **different** values for the limit along different ways of approaching $(0,0)$ the limit **DNE** (Does Not Exist).

**Second Way**. Let's see what is going along a typical line with slope $m$:

$$\lim_{(x,y)\to(0,0),y=mx} \frac{x^2+2(mx)^2}{x^2+3(mx)^2} = \lim_{x\to 0} \frac{(1+2m^2)x^2}{(1+3m^2)x^2} = \lim_{x\to 0} \frac{(1+2m^2)}{(1+3m^2)} = \frac{(1+2m^2)}{(1+3m^2)} \quad .$$

This this **depends** on $m$, the limit **DNE**.

**8.** (10 points) Find the length of the curve

$$\mathbf{r}(t) = \langle\, t \,,\, 2\sin t \,,\, 2\cos t \,\rangle \quad 0 \le t \le 3 \quad .$$

**Sol.**
The **type** of the answer is: **number**.

The relevant forumula is

$$ArcLength = \int_{t_0}^{t_1} |\mathbf{r}'(t)|\, dt \quad .$$

We have

$$\mathbf{r}'(t) = \langle\, 1 \,,\, 2\cos t \,,\, -2\sin t \,\rangle \quad ,$$

So

$$|\mathbf{r}'(t)| = \sqrt{1^2 + (2\cos t)^2 + (-2\sin t)^2} =$$

$$\sqrt{1^2 + 4\cos^2 t + 4\sin^2 t} = \sqrt{1^2 + 4(\cos^2 t + \sin^2 t)} = \sqrt{1^2+4} = \sqrt{5} \quad .$$

So we have

$$ArcLength = \int_0^3 \sqrt{5}\, dt = \sqrt{5}\int_0^3 dt = \sqrt{5}\cdot 3 = 3\sqrt{5} \quad .$$

**Ans.** $3\sqrt{5}$.

**9.** (10 points) A certain particle has acceleration

$$\mathbf{a}(t) = \langle\, 12t^2 \,,\, 20t^3 \,,\, 30t^4 \,\rangle \quad ,$$

and at $t = 1$ its velocity is $\langle 4, 5, 6\rangle$ and its position vector is $\langle 1, 1, 1\rangle$, find its velocity and position vector at time $t = 2$.

**Sol.**
The **type** of the answer is: **vectors of numbers**.

$$\mathbf{v}(t) = \int \langle\, 12t^2 \,,\, 20t^3 \,,\, 30t^4 \,\rangle \, dt = \langle\, 4t^3 \,,\, 5t^4 \,,\, 6t^5 \,\rangle + \mathbf{C} \quad .$$

$$\mathbf{v}(1) = \langle\, 4 \,,\, 5 \,,\, 6 \,\rangle + \mathbf{C}$$

On the other hand, from the problem $\mathbf{v}(1) = \langle\, 4 \,,\, 5 \,,\, 6 \,\rangle$, so $\mathbf{C} = \mathbf{0}$, and we have

$$\mathbf{v}(t) = \langle\, 4t^3 \,,\, 5t^4 \,,\, 6t^5 \,\rangle \quad .$$

**Next**:

$$\mathbf{r}(t) = \int \langle\, 4t^3 \,,\, 5t^4 \,,\, 6t^5 \,\rangle \, dt = \langle\, t^4 \,,\, t^5 \,,\, t^6 \,\rangle \ + \mathbf{C} \quad .$$

Once again $\mathbf{C} = \mathbf{0}$, so

$$\mathbf{r}(t) = \langle\, t^4 \,,\, t^5 \,,\, t^6 \,\rangle \quad .$$

**Finally**, plugging-in $t = 2$, we get

$$\mathbf{v}(2) = \langle\, 4 \cdot 2^3 \,,\, 5 \cdot 2^4 \,,\, 6 \cdot 2^5 \,\rangle = \langle\, 32 \,,\, 80 \,,\, 192 \,\rangle$$

$$\mathbf{r}(2) = \langle\, 2^4 \,,\, 2^5 \,,\, 2^6 \,\rangle = \langle\, 16 \,,\, 32 \,,\, 64 \,\rangle$$

**Ans.**: The velocity when the time was $t = 2$ was $\langle\, 32 \,,\, 80 \,,\, 192 \,\rangle$ and the position vector was $\langle\, 16 \,,\, 32 \,,\, 64 \,\rangle$

**10.** (10 points) Find an equation to the plane that passes through the points $(1, 1, 1)$, $(2, 2, 0)$, $(0, 3, 0)$.

**Sol.**
The **type** of the answer is: Equation of a plane.

Let's call $P = (1, 1, 1)$, $Q = (2, 2, 0)$, $R = (0, 3, 0)$,

$$\mathbf{PQ} = Q - P = \langle 2-1, 2-1, 0-1 \rangle = \langle 1, 1, -1 \rangle \quad ,$$

$$\mathbf{PR} = R - P = \langle 0-1, 3-1, 0-1 \rangle = \langle -1, 2, -1 \rangle \quad .$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -1 \\ -1 & 2 & -1 \end{vmatrix} \quad =$$

$$\mathbf{i} \begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -1 \\ -1 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}$$

$$= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k} = \langle 1, 2, 3 \rangle \quad .$$

This is the **normal vector** $\mathbf{n} = \langle a, b, c \rangle$. So $\mathbf{n} = \langle a, b, c \rangle = \langle 1, 2, 3 \rangle$. The equation is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \quad .$$

where $(x_0, y_0, z_0)$ is **any** of the three points (doesn't matter which). Picking $(1, 1, 1)$ we get:

$$1(x - 1) + 2(y - 1) + 3(z - 1) = 0 \quad ,$$

and doing the algebra, we get:

$$x + 2y + 3z = 6 \quad .$$

**Ans.** $x + 2y + 3z = 6$ .
**Comment**: TO check your answer you should plug-in each of the three points into the equation and see if you get a right statement.
$(1, 1, 1)$: $1 + 2 \cdot 1 + 3 \cdot 1 = 6$ , OK!
$(2, 2, 0)$: $2 + 2 \cdot 2 + 0 \cdot 1 = 6$ , OK!
$(0, 3, 0)$: $0 + 2 \cdot 3 + 0 \cdot 1 = 6$ , OK!