Dr. Z's Math251 Handout #17.3 (2nd ed.) [The Divergence Theorem]

By Doron Zeilberger

Problem Type 17.3a: Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\mathbf{k} \quad ,$$

where S is a surface bounding some solid region that can be expressed in some coordinate system (Cartesian, cylindrical, or spherical).

Example Problem 17.3a: Use the Divergence Theorem to calculate the surface integral $\int\int_S \mathbf{F} \cdot d\,\mathbf{S}$, where

$$\mathbf{F}(x,y,z) = e^{2x}\sin 2y\,\mathbf{i} + e^{2x}\cos 2y\,\mathbf{j} + y^2z^2\mathbf{k} \quad ,$$

where S is the surface of the box bounded by the planes $x = 0$, $x = 2$, $y = 0$, $y = 1$, $z = 0$, $z = 3$.

Steps

1. Compute the divergence of $\mathbf{F}$:

$$div\,\mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad .$$

2. Write the solid region E in 'iterated form' in either Cartesian, Cylindrical, or Spherical coordinates.

Example

1.

$$div\,\mathbf{F} = \frac{\partial}{\partial x}(e^{2x}\sin 2y) + \frac{\partial}{\partial y}(e^{2x}\cos 2y) + \frac{\partial}{\partial z}(y^2z^2)$$

$$= 2e^{2x}\sin 2y - 2e^{2x}\sin 2y + 2y^2z = 2y^2z \quad .$$

2. The solid region E bounding our surface is clearly the box

$$\{\,(x,y,z)\,|\,0 \le x \le 2\,,\,0 \le y \le 1\,,\,0 \le z \le 3\,\} \quad .$$

3. Set-up the Divergence Theorem, and evaluate the triple integral.

$$\int\int_S \mathbf{F} \cdot d\,\mathbf{S} = \int\int\int_E div\,\mathbf{F}\,dV \quad .$$

3.

$$\int\int_S \mathbf{F} \cdot d\,\mathbf{S} = \int\int\int_E div\,\mathbf{F}\,dV \quad .$$

$$= \int_0^2 \int_0^1 \int_0^3 2y^2 z\,dz\,dy\,dx \quad .$$

$$= 2\left(\int_0^2 dx\right)\left(\int_0^1 y^2\,dy\right)\left(\int_0^3 z\,dz\right)$$

$$= 2(2-0)\cdot\frac{1^3-0^3}{3}\cdot\frac{3^2-0^2}{2} = 2\cdot 3 = 6 \quad .$$

Ans.: 6.

A Problem from a previous Final

Let

$$F(x,y,z) =$$

$$\langle\, \cos(\sqrt{1+x} + zy^3) \quad , \quad \tan(x+y^2+1/z) \quad , \quad \tan^{-1}(e^{xyz} + \cos(x^2 - y + 3z)\,\rangle \quad ,$$

and let $\langle P, Q, R \rangle = curl\mathbf{F}$. Compute

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \quad .$$

Be sure to explain everything.

Ans.: 0 (taking the divergence of curl is always 0).