Dr. Z's Math251 Handout #16.5 (2nd ed.)
[Surface Integrals of Functions and Surface Integrals of Vector Fields]

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Problem Type 16.5a: Evaluate the surface integral

$$\int\int_S F(x,y,z)\,dS \quad ,$$

where S is a given surface that is either given parametrically in terms of u v, or can be made so, or can be written in the form $z = f(x,y)$, $\{(x,y)|,(x,y) \in D\}$ for some set D in the xy-plane.

Example Problem 16.5a: Evaluate the surface integral

$$\int\int_S 2x^2z^2\,dS \quad ,$$

where S is the part of the cone $z^2 = x^2 + y^2$ that lies between the planes $z = 1$ and $z = 3$.

Steps **Example**

1. If the surface can be described parametrically as

$$\{(x(u,v), y(u,x), z(u,v)) | (u,v) \in D\} \quad ,$$

for some set D in uv-plane, set-up $r = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, and compute r_u, r_v, then their cross product $r_u \times r_v$, and finally its length $|r_u \times r_v|$. Put

$$dS = |r_u \times r_v|\, du\, dv \quad .$$

On the other hand if the surface is given as $z = f(x,y)$, with some description where it lives, figure out the "floor" (projection on the xy-plane), and put

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$

Note: The first method also works in the second case, just take x, y as the parameters and put $x = x, y = y, z = f(x,y)$.

1. In this problem $z^2 = x^2 + y^2$ means $z = \sqrt{x^2+y^2}$, so it is possible to do it the second way. But it is a bit easier to use cylindrical coordinates and then the surface is $z = r$, and the parametric representation is

$$x = r\cos\theta\,,\ y = r\sin\theta\,,\ z = r \quad ,$$

where the parameters are r and θ. The region on the xy-plane below the surface is

$$\{(r,\theta)\,|\,1 \le r \le 3, 0 \le \theta \le 2\pi\} \quad .$$

So

$$\mathbf{r}(r,\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\,\mathbf{k} \quad .$$

$$\mathbf{r}_r = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + \,\mathbf{k} \quad .$$

$$\mathbf{r}_\theta = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j} + 0\,\mathbf{k} \quad .$$

$$\mathbf{r}_r \times \mathbf{r}_\theta = -r\cos\theta\,\mathbf{i} - r\sin\theta\,\mathbf{j} + r\,\mathbf{k} \quad ,$$

(you do it!). Also

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{(-r\cos\theta)^2 + (-r\sin\theta)^2 + r^2} = \sqrt{2}r \quad ,$$

so $dS = \sqrt{2}r\,dr\,d\theta$.

2. Set-up the integral

$$\int\int_S F(x,y,z)\,dS \quad ,$$

with the x, y, z replaced by their expressions in terms of the parameters, and S replaced by its description in terms of u, v, and dS replaced by what you found in step 1.

2.

$$\int\int_S 2x^2 z^2\,dS$$

$$= \int\int_D 2(r\cos\theta)^2 r^2 \sqrt{2} r\,dr\,d\theta \quad ,$$

where D is the region $1 \le r \le 3$, $0 \le \theta \le 2\pi$. Converting it to an iterated integral, we get

$$= 2\sqrt{2}\int_0^{2\pi}\int_1^3 r^5\cos^2\theta\,dr\,d\theta \quad .$$

3. Evaluate the iterated integral.

3.

$$= 2\sqrt{2}\int_0^{2\pi}\int_1^3 r^5\cos^2\theta\,dr\,d\theta \quad .$$

$$= \sqrt{2}\int_0^{2\pi}\left[\int_1^3 r^5 2\cos^2\theta\,dr\right]\,d\theta \quad .$$

$$= \sqrt{2}\int_0^{2\pi} 2\cos^2\theta\left[\frac{r^6}{6}\Big|_1^3\right]\,d\theta \quad .$$

$$= \sqrt{2}\left(\frac{3^6-1^6}{6}\right)\cdot\int_0^{2\pi} 2\cos^2\theta d\theta$$

$$= \sqrt{2}\frac{364}{3}\cdot\int_0^{2\pi}(1+\cos(2\theta))d\theta$$

$$= \sqrt{2}\left(\frac{364}{3}\right)\cdot(\theta+\frac{\sin(2\theta)}{2})\Big|_0^{2\pi} = \sqrt{2}\,\frac{364}{3}\cdot 2\pi = \frac{728\sqrt{2}\pi}{3} \quad .$$

Ans.: $\frac{728\sqrt{2}\pi}{3}$.

Problem Type 16.5b: Evaluate the surface integral $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$ and oriented surface S.

$$\mathbf{F}(x,y,z) = P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k} \quad ,$$

where S is the part of the surface $z = g(x,y)$ that lies above some region D, in the xy-plane and has upward orientation.

Example Problem 16.5b: Evaluate the surface integral $\int\int_S \mathbf{F}\cdot d\mathbf{S}$ for the given vector field $\mathbf{F}$

and oriented surface S.

$$\mathbf{F}(x,y,z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k} \quad ,$$

S is the part of the paraboloid $z = 4 - x^2 - y^2$ that lies above the square $0 \le x \le 1, 0 \le y \le 1$ and has upward orientation.

Steps

1. Set-up the formula

$$\int\int_S \mathbf{F} \cdot d\mathbf{S} =$$

$$\int\int_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R \right) dA$$

Be also sure to replace z by $g(x,y)$.

Example

1. Here $g = 4 - x^2 - y^2$,

$$P = xy \quad , \quad Q = yz \quad , \quad R = zx \quad .$$

Plugging everything in, our surface integral

$$= \int\int_D -xy(-2x) - yz(-2y) + xz)\,dA$$

$$= \int\int_D (2x^2y + (2y^2 + x)z)\,dA \quad ,$$

but since $z = 4 - x^2 - y^2$, this equals

$$= \int\int_D (2x^2y + (2y^2 + x)(4 - x^2 - y^2))\,dA \quad ,$$

$$= \int\int_D (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2)\,dA \quad ,$$

where D is the square

$$\{\,(x,y)\,|\,0 \le x \le 1\,,\,0 \le y \le 1\} \quad .$$

2. Looking at D convert it into an iterated integral.

2.

$$= \int_0^1 \int_0^1 (2x^2y + 8y^2 - 2y^2x^2 - 2y^4 + 4x - x^3 - xy^2)\,dx\,dy \quad .$$

$$= \int_0^1 \int_0^1 (-x^3 + x^2(-2y^2 + 2y) + x(-y^2 + 4) - 2y^4 + 8y^2)\,dx\,dy \quad .$$

3. Evaluate the iterated integral.

3. You do it! (it is rather tedious). The answer turns out to be $\frac{713}{180}$.

A Problem from a Previous Final: Evaluate the surface integral

$$\int\int_S \sqrt{3}\, x \, dS \quad ,$$

where S is the triangular region with vertices $(1,0,0), (0,1,0), (0,0,1)$.

Ans. $\frac{1}{2}$.