Dr. Z's Math251 Handout #16.2 (2nd ed.) [Line Integrals]

By Doron Zeilberger

Problem Type 16.2a: Evaluate the line integral,

$$\int_C f(x,y)\,ds \quad ,$$

where C is some curve that the problem gives you in parametric form, or you have to represent yourself (typically circles, line-segments, semicircles etc.).

Example Problem 16.2a: Evaluate the line integral,

$$\int_C x^2 y\,ds \quad ,$$

where C is top half of the circle $x^2 + y^2 = 9$.

Steps

1. Find the parametric equation of the curve $(x(t), y(t))$, $a \le t \le b$, unless it is given by the problem.

Example

1. The parametric equation of a circle of the form $x^2 + y^2 = r^2$ is

$$x = r\cos t\,,\ y = r\sin t \quad .$$

So in our case we have $r = 3$ and

$$x = 3\cos t\,,\ y = 3\sin t \quad .$$

Since it is the *top* half, t goes from 0 to π, so $0 \le t \le \pi$.

Steps

2. Compute

$$\sqrt{x'(t)^2 + y'(t)^2} \quad .$$

Example

2. $x'(t) = -3\sin t$, $y'(t) = 3\cos t$, so

$$\sqrt{x'(t)^2 + y'(t)^2} = \sqrt{(-3\sin t)^2 + (3\cos t)^2}$$

$$= \sqrt{9\sin^2 t + 9\cos^2 t} = \sqrt{9(\sin^2 t + \cos^2 t)} = \sqrt{9} = 3 \quad .$$

3. The line integral is

$$\int_a^b f(x(t), y(t))\sqrt{x'(t)^2 + y'(t)^2}\, dt \quad .$$

Convert everything to the t-language and evaluate the t-integral from $t = a$ to $t = b$.

3.

$$\int_C x^2 y\, ds =$$

$$\int_0^\pi (3\cos t)^2\, (3\sin t) \cdot 3\, dt$$

$$81 \int_0^\pi \cos^2 t\, \sin t\, dt = 81 \left(\frac{-\cos^3 t}{3} \Big|_0^\pi \right)$$

$$= (-27)(\cos^3 \pi - \cos^3 0) = 54 \quad .$$

Ans.: 54.

Problem Type 16.2b: Evaluate the line integral

$$\int_C P(x, y, z)\, dx + Q(x, y, z) dy + R(x, y, z)\, dz \quad ,$$

where $C : x = x(t)\,,\ y = y(t)\,,\ z = z(t)\,,\ a \le t \le b$.

Example Problem 16.2b: Evaluate the line integral

$$\int_C y\, dx + x\, dy + x^2 y \sqrt{z}\, dz \quad ,$$

where $C : x = t^3\,,\ y = t\,,\ z = t^2\,,\ 0 \le t \le 1$.

Steps

1. Get a (single variable) definite integral, in t, from $t = a$ to $t = b$, by changing x, y, z to their expressions in terms of t and dx, dy, dz to $x'(t)dt$, $y'(t)dt$, $z'(t)dt$, respectively,

$$\int_C P(x, y, z)\, dx + Q(x, y, z) dy + R(x, y, z)\, dz \quad .$$

$$= \int_a^b [P(x(t), y(t), z(t))x'(t) +$$

$$Q(x(t), y(t), z(t))y'(t) +$$

$$R(x(t), y(t), z(t))z'(t)]\, dt \quad .$$

Example

1.

$$\int_C y\, dx + x\, dy + x^2 y \sqrt{z}\, dz$$

$$= \int_0^1 t(3t^2) dt + t^3 dt + (t^3)^2 t \sqrt{t^2} (2t) dt$$

$$= \int_0^1 [4t^3 + 2t^9] dt \quad .$$

2. Evaluate the t-integration.

2.

$$= t^4 + \frac{t^{10}}{5}\Big|_0^1 =$$

$$= 1 + \frac{1}{5} - 0 = \frac{6}{5} \quad .$$

Ans.: $\frac{6}{5}$.

Problem Type 16.2c: Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x,y,z) = P(x,y,z)\mathbf{i} + Q(x,y,z)\mathbf{j} + R(x,y,z)\mathbf{k} \quad ,$$

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k} \quad , \quad a \le t \le b \quad .$$

Example Problem 16.2c: Evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r} \quad ,$$

where C is given by the vector function $\mathbf{r}(t)$.

$$\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k} \quad ,$$

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \quad , \quad 0 \le t \le 2 \quad .$$

Steps

1. The desired line-integral equals

$$\int_C P\,dx + Q\,dy + R\,dz \quad .$$

Set-it up.

Example

1. Our integral is

$$\int_C yz\,dx + xz\,dy + xy\,dz \quad ,$$

where $x = t$, $y = t^2$, $z = t^3$, $0 \le t \le 2$.

2. Evaluate this line integral like we did above (16.2b).

2.

$$= \int_0^2 (t^2)(t^3)\,dt + (t)(t^3)(2t)\,dt + (t)(t^2)(3t^2)\,dt \quad ,$$

$$= \int_0^2 [t^5 + 2t^5 + 3t^5]\,dt$$

$$= \int_0^2 6t^5\,dt = t^6\Big|_0^2 = 2^6 - 0^6 = 64 \quad .$$

Ans.: 64.

A Problem from a previous Final

Let C be the line segment from $(0,1)$ to $(3,5)$, find $\int_C 2xy\,ds$.

Ans.: 55.

Another Problem from a Previous Final

(a) (4 points) Compute the surface integral

$$\int\int_S 8\,dS \quad ,$$

where S is the sphere $(x-1)^2 + (y+4)^2 + (z-9)^2 = 100$.

(b) (4 points) Compute the triple integral

$$\int\int\int_E 30\,dV \quad ,$$

where E is the ball $\{\,(x,y,z) \,|\, (x-1)^2 + (y+4)^2 + (z-9)^2 \leq 100\,\}$.

(c) (4 points) Compute the line integral

$$\int_C 3\,ds \quad ,$$

where C is the circumference of the region $\{(x,y) \,|\, x^2 + y^2 \leq 4\,,\, y \geq 0\}$.

Ans.: 3200π, 40000π, $6\pi + 12$.